

# THE HEISENBERG GROUP AND CONFORMAL FIELD THEORY

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ABSTRACT. A mathematical construction of the conformal field theory (CFT) associated to a compact torus, also called the “nonlinear  $\sigma$ -model” or “lattice-CFT”, is given. Underlying this approach to CFT is a unitary modular functor, the construction of which follows from a “Quantization commutes with reduction”-type theorem for unitary quantizations of the moduli spaces of holomorphic torus-bundles and actions of loop groups. This theorem in turn is a consequence of general constructions in the category of affine symplectic manifolds and their associated generalized Heisenberg groups.

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## INTRODUCTION

The aim of this paper is to give a construction of certain conformal field theories associated to a compact Abelian Lie group  $T$  using the representation theory of the associated loop group. In physics terminology this conformal field theory is called the nonlinear  $\sigma$ -model with target  $T$ , in the vertex algebra literature it is usually called the lattice-model associated to  $\pi_1(T)$ . It is an Abelian version of the WZW-model which describes strings moving on an arbitrary compact Lie group.

For this we use the mathematical axiomatisation of conformal field theory given by Graeme Segal in [S3]. This approach to conformal field theory can be paraphrased by the statement that a conformal field theory is nothing but a projective representation of the two-dimensional complex cobordism category. Despite the beauty and transparency of this definition, so far not many examples of this structure have been rigorously constructed. To the author's knowledge, so far only the chiral case of fermions on Riemann surfaces, and its spin versions, see [S3, K], is known to exist. The present paper adds another example.

Although arguably much simpler than the non-Abelian WZW-model, the Abelian case is interesting enough in the sense that it is an example of a *non-chiral* conformal field theory based on a higher dimensional modular functor. To wit, the above mentioned chiral conformal field theory of fermions is given by a one-dimensional modular functor. On the other hand, the case of a torus stands out—in comparison with the non-Abelian case—due to the presence of an important extra symmetry structure given by the Heisenberg group, and it is exactly this feature that we exploit in this paper. Indeed the moduli space of flat  $T$ -bundles over a closed surface is an Abelian variety, and therefore its geometric Quantization carries an irreducible representation of a Heisenberg extension of a finite group, a fact which goes back to the fundamental papers [Mu1]. Since we need to address gluing of surfaces, we shall introduce surfaces with boundaries. On the level of moduli spaces, this leads to an infinite dimensional version of the theory of Abelian varieties, coined “affine symplectic manifolds”, whose Quantization corresponds to representations of certain associated infinite dimensional Heisenberg groups. In the infinite dimensional case, the concept of a polarization of such a variety plays a crucial role.

In this approach to conformal field theory, unitarity is of the utmost importance. Indeed its construction follows from proving that a certain well-known modular functor is unitary. This is the reason why in this paper we work exclusively with Hilbert spaces and (projective) unitary representations. Concretely, we realize the spaces of conformal blocks of the modular functor as multiplicity spaces of certain positive energy representations of loop groups associated to Riemann surfaces. Factorization, in a unitary fashion, then follows from “gluing laws” for these representations, which in turn can be interpreted as a “Quantization commutes with reduction”-type of theorem.

Finally, a word about the literature: geometric Quantization of the finite dimensional Abelian moduli space and its connection with the theory of Theta-functions is briefly discussed in [At]. In particular, here it is mentioned that the projective flatness of the resulting Quantization can be derived as a “cohomological rigidity” using the representation theory of the associated finite Heisenberg group, see also [R]. This idea plays a central role in the present paper. From the point of view of conformal field theory, the abelian, or lattice, case is discussed in [S3]—see also

the “new” introduction to this paper–, and [HK]. Remark that the construction of the modular functor in the present paper, namely via the moduli spaces of flat connections, is quite different; the connection is given in §3.4. Finally, in [An], the relation between geometric and deformation Quantization of the abelian moduli spaces is discussed.

**Outline of the paper.** This paper is organized in the following way: the first section is devoted to affine symplectic manifolds, their quantization and reduction. In section 2 it is proved that the moduli space of flat  $T$ -bundles fits into this scheme, which allows us to quantize them to positive energy representations of  $LT$ , and use reduction to provide “gluing laws” under sewing of surfaces. This leads in section 3 to the construction of a unitary modular functor, which in turn is used to construct the conformal field theory. Finally, we give a short outline of the proof of the (well-known) theorem that the category of positive energy representation of  $LT$  is a modular tensor category.

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## 1. QUANTIZATION OF AFFINE SYMPLECTIC MANIFOLDS

**1.1. Affine symplectic manifolds.** In this section we describe a certain category of symplectic manifolds, possibly infinite dimensional, which can be quantized in a rather straightforward way. This defines the appropriate framework from which we approach the moduli space of flat Abelian connections in Section 2. Let us remark from the outset that all infinite dimensional manifolds in this paper will be modelled on complete nuclear topological vector spaces.

**Definition 1.1.** An affine symplectic manifold  $(X, \omega)$  is a weakly symplectic manifold modelled on  $V$  which carries a symplectic  $V$ -action whose isotropy groups  $V_x$ ,  $x \in X$  are finitely generated lattices.

Thus  $T_x X$  can be identified with  $V$  for each  $x \in X$ , and  $V_x \cong \pi_1(X, x)$ . It is easy to see that the symplectic action of  $V$  on  $X$  furnishes  $V$  with a weak linear symplectic form, also denoted  $\omega$ . Remark that, after choosing a basepoint  $x_0 \in X$ , an affine symplectic manifold is just an (infinite dimensional) Abelian group. Acting with  $V$  on the basepoint, an affine symplectic manifold fits into an exact sequence of Abelian groups

$$(1.1) \quad 0 \longrightarrow \pi_1(X, x_0) \longrightarrow V \xrightarrow{\pi} X \longrightarrow \pi_0(X, x_0) \longrightarrow 0.$$

In the following, we will freely choose a basepoint although all constructions are actually (up to isomorphism) independent of such a choice. Notice that in the main examples of this paper, moduli spaces of flat bundles, there is in fact a canonical basepoint, viz. the class of the trivial bundle.

If the symplectic form  $\omega$  represents an integral class in  $H^2(X)$ , i.e., if  $\omega|_{V_{x_0} \times V_{x_0}}$  is integral, then we can find hermitian line bundles  $(L, h)$  on  $X$  with unitary connections  $\nabla$  which have curvature  $-i\omega$ . In the language of geometric Quantization, such a line bundle is called a prequantum line bundle. The set of prequantum

line bundles forms a torsor for the finite dimensional torus  $H^1(X, \mathbb{T})$  of flat line bundles.

Given such an  $L$ , we have a homomorphism  $V \rightarrow H^1(X, \mathbb{T})$  given by

$$v \mapsto [L^* \otimes v^* L].$$

Let  $V_X$  be its kernel. (Note that  $\pi_0(V_X) \cong H^1(X, \mathbb{Z}) \cong \text{Hom}(V_{x_0}, \mathbb{Z})$ .) Define

$$\tilde{V}_X := \left\{ \begin{array}{l} \text{connection-preserving bundle automorphisms of } (L, \nabla, h) \\ \text{which cover the action of an element of } V_X \text{ on } X \end{array} \right\}.$$

**Proposition 1.2.** *The group  $\tilde{V}_X$  is a central extension*

$$0 \rightarrow H^0(X, \mathbb{T}) \rightarrow \tilde{V}_X \rightarrow V_X \rightarrow 0.$$

When  $X$  is connected, this is the (degenerate) Heisenberg group of  $(V_X, \omega|_{V_X \times V_X})$  with center  $\tilde{V}_{x_0}$ .

The proof of this proposition is obvious. With hindsight notice the following:

**Proposition 1.3.** *For  $X$  connected, there is an equivalence of categories:*

$$\left\{ \begin{array}{l} \text{Prequantum line bundles} \\ \text{over } (X, \omega). \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Splittings of the extension } \tilde{V}_{x_0} \\ \text{induced from the Heisenberg group } \tilde{V}. \end{array} \right\}$$

Remark that  $V_{x_0}$  is an isotropic lattice in  $(V, \omega)$ . The functor establishing the equivalence above sends a splitting  $\psi$ , viewed as a character  $\psi : \tilde{V}_{x_0} \rightarrow \mathbb{T}$ , to the line bundle

$$L_\psi := \tilde{V} \times_{\tilde{V}_{x_0}} \mathbb{C}_\psi.$$

Conversely, the fiber over  $x_0$  of a prequantum line bundle carries a unitary representation of  $\tilde{V}_{x_0}$ , i.e., determines a character  $\psi : \tilde{V}_{x_0} \rightarrow \mathbb{T}$ . Also notice that the set of splittings of  $\tilde{V}_{x_0}$  forms a torsor over the Pontryagin dual  $\hat{V}_{x_0} := \text{Hom}(V_{x_0}, \mathbb{T})$ , which corresponds to the action of the torus of flat line bundles using the isomorphism  $\text{Hom}(V_{x_0}, \mathbb{T}) \cong H^1(X, \mathbb{T})$ .

When  $X$  is not connected,  $\tilde{V}_X$  is not a central extension by  $\mathbb{T}$  as it contains the extension of the group of components  $\pi_0(X) \subset V_X$  which is canonically split. To cure this, one simply adds its Pontryagin dual to define

$$A(X) := V_X \times \widehat{\pi_0(X)}.$$

Including the dual has the effect that now  $A(X)$  has a canonical Heisenberg central extension

$$1 \rightarrow \mathbb{T} \rightarrow \widehat{A(X)} \rightarrow A(X) \rightarrow 1$$

by combining the extension  $\tilde{V}_X$  of Proposition 1.2, which is canonically trivial over  $\pi_0(X)$ , with the unique Heisenberg group associated to the canonical 2-cocycle

$$\psi : \pi_0(X) \times \widehat{\pi_0(X)} \rightarrow \mathbb{T}$$

given by

$$(1.2) \quad \psi((\lambda_1, \alpha_1), (\lambda_2, \alpha_2)) = \alpha_2(\lambda_1) \alpha_1(\lambda_2)^{-1}.$$

By definition,  $\tilde{V}_X$  acts on  $L$  covering the action of  $V$ . There is a unique way to make it  $\widehat{A(X)}$ -equivariant by having  $\widehat{\pi_0(X)}$  act by  $\lambda \in \pi_0(X)$  on the connected component labeled by  $\lambda$ . The very nature of the cocycle (1.2) implies that this defines an action of  $A(X)$ .

**Definition 1.4.** A polarization of an affine symplectic manifold  $X$  modelled on  $V$  means a polarization of  $V$ .

A polarization of a symplectic vector space is a positive compatible complex structure, as explained in the appendix, cf. Definition A.2. A polarized affine symplectic manifold is simply an affine symplectic manifold together with the choice of a polarization. Notice that when the manifold is a finite dimensional symplectic torus, this notion coincides with that of a polarized Abelian variety.

**1.2. Quantization.** Assume for a moment that the affine symplectic manifold  $X$  is finite dimensional. The way to proceed in geometric Quantization is to define, for each polarization, the Hilbert space  $H_{L^2}^0(X; L) := L_{hol}^2(X, L)$  of holomorphic sections of a prequantum line bundle  $(L, h)$  with inner product given by

$$\langle s_1, s_2 \rangle := \int_X h(s_1, s_2) \frac{\omega^n}{n!},$$

where  $\dim X = 2n$ . Going over to the infinite dimensional case, one immediately realizes that the Liouville measure doesn't exist as there are no nontrivial invariant measures on an infinite dimensional vector space. However, below we will show that for affine symplectic manifolds, the combination " $h(-, -)\omega^n/n!$ " does make sense as a line bundle valued measure (cf. [Pi], and see below). The rationale behind this is that for an infinite dimensional vector space, this combination formally combines into a Gaussian measure which can be rigorously constructed in infinite dimensions.

Let  $(X, L)$  be a polarized affine symplectic manifold modelled on  $V$ . First of all, a thickening of  $X$  is a smooth manifold  $X^*$  modelled on  $V^*$  equipped with a continuous dense embedding  $X \hookrightarrow X^*$ . More precisely, the polarization  $J$  puts  $V$  and  $V^*$  in a triple  $V \subseteq H_J \subseteq V^*$ , where  $H_J$  is an intermediate Hilbert space, the completion of  $V$  in the inner product defined by  $\omega$  and  $J$ , cf. Appendix A. Therefore, we aim for a "rigged manifold"

$$X \hookrightarrow X_H \hookrightarrow X^*,$$

where the intermediate space  $X_H$  is a Hilbert manifold modelled on  $H_J$ . We will prove below that there exists a canonical thickening of any affine symplectic manifold and that the line bundle  $L$  extends to  $X^*$ . Then, a measure  $\mu^L$  on  $X^*$  with values in  $L$  consists of a continuous map

$$\mu^L : \Gamma(X^*, L) \times \Gamma(X^*, L) \rightarrow \{\text{complex Borel measures on } X^*\},$$

written  $(s_1, s_2) \mapsto \mu_{s_1, s_2}^L$ , or  $h(s_1, s_2)d\mu$ , satisfying the following properties:

- for all  $f_1, f_2 \in C(X)$  we have

$$\mu_{f_1 s_1, f_2 s_2}^L = \bar{f}_1 f_2 \mu_{s_1, s_2}^L,$$

i.e., the map is sesquilinear as a map of  $C(X)$ -modules,

- the measure is positive in the sense that

$$\mu_{s, s}^L \geq 0,$$

for all sections  $s \in \Gamma(X^*, L)$ , with strict inequality for  $s \neq 0$ ,

- for  $s_i \in \Gamma(X^*, L)$ ,  $i = 1, \dots, 4$ , the measure  $\mu_{s_1, s_2}^L$ , restricted to the set  $\{x \in X^*, h_L(s_3(x), s_4(x)) \neq 0\}$ , is absolutely continuous with respect to  $\mu_{s_3, s_4}^L$  with Radon–Nikodym derivative equal to

$$\frac{d\mu_{s_1, s_2}^L}{d\mu_{s_3, s_4}^L} = \frac{h(s_1, s_2)}{h(s_3, s_4)}.$$

**Proposition 1.5.** *An affine symplectic manifold  $X$  has a canonical thickening  $X^*$  such that any prequantum line bundle  $L$  extends to  $X^*$ , and a polarization  $J$  of  $X$  defines a unique  $\widehat{A}(X)$ -invariant measure  $\mu_J^L$  on  $X^*$  with values in  $L$ .*

*Proof.* First consider the statement for a symplectic vector space  $(V, \omega)$ . The, up to isomorphism, unique prequantum line bundle may be realized as the trivial bundle  $L = V \times \mathbb{C}$  equipped with the Hermitian metric

$$h((v, z_1), (v, z_2)) = e^{-\langle v, v \rangle / 2} \bar{z}_1 z_2,$$

where  $\langle \cdot, \cdot \rangle$  is the metric defined by the polarization  $J$ , cf. (A.2). Indeed one computes  $-\partial\bar{\partial} \log h = -i\omega$ . Therefore, the action of  $V$  by translations lifts to connection preserving transformations of  $V \times \mathbb{C}$  by the formula

$$(1.3) \quad v \cdot (w, z) = (v + w, e^{\langle v, v \rangle / 4 + \langle v, w \rangle / 2} z).$$

This defines an action of the Heisenberg group  $\tilde{V}$  associated to the symplectic vector space  $(V, \omega)$ ; this gives an explicit realization of Proposition 1.2 above.

As described in Appendix A.2, a compatible positive complex structure  $J$  on  $(V, \omega)$  defines a family of Gaussian measures  $\mu_J^t$ ,  $t > 0$ , on  $V^*$ , the dual of  $V$ , together with a dense embedding  $V \hookrightarrow V^*$ . Put  $\mu_J := \mu_J^1$ , extend the trivial bundle  $L$  to  $V^*$ , and define, for a section  $F$  of  $L$  over  $V^*$ , i.e., a continuous function  $F : V^* \rightarrow \mathbb{C}$ ;

$$d\mu_{F_1, F_2}^L := \bar{F}_1 F_2 d\mu_J.$$

It is obvious that when  $F$  is nonzero, this defines a positive Borel measure. Acting by  $\tilde{V}$  as induced from (1.3), this measure is invariant since the exponential factors in  $\bar{F}F$  cancel against the contribution of the Cameron–Martin formula (A.3). We thus observe that in the linear case, the Proposition is merely a reformulation of the existence and properties of the Gaussian measure, which indeed is uniquely determined by the polarization  $J$ .

When  $V$  is finite dimensional, the Gaussian measure thus defined can of course be written as the product

$$d\mu_J(v) = e^{-\langle v, v \rangle / 2} \frac{\omega^n}{n!},$$

where  $\omega^n / n!$  is the Liouville measure, i.e., a Haar measure on  $V$ . In fact this is the expression for any finite dimensional affine symplectic manifold: the  $L$ -valued measure is defined as

$$\mu_{s_1, s_2}^L := h(s_1, s_2) \frac{\omega^n}{n!},$$

where  $h$  is the Hermitian metric on  $L$ . Finally, in the general case, notice that a polarization  $J$  induces a metric on the modelling space  $V$ , which decomposes  $X = W \times Y$ , with  $W$  a symplectic vector space and  $Y$  a finite dimensional affine symplectic manifold. This reduces the general statement to the two special cases

considered above, and the resulting measure is independent of the decomposition. By construction, this line bundle valued measure is invariant.  $\square$

With the help of this measure one can define the Quantization of  $X$  to be the  $L^2$ -space of holomorphic sections of  $L$  on  $X^*$  with the inner-product given by

$$\langle s_1, s_2 \rangle := \int_{X^*} h(s_1, s_2) d\mu.$$

We denote this Hilbert space by  $\mathcal{H}_X := L^2_{\text{hol}}(X, L)$ . When  $X$  is disconnected, this Hilbert space is defined as

$$\mathcal{H}_X = \bigoplus_{\lambda \in \pi_0(X)} \mathcal{H}_{X, \lambda},$$

where the direct sum is taken in the  $L^2$ -sense. When we want to stress the dependence on the line bundle  $L$  in the notation, we write  $\mathcal{H}_{X, L}$  instead of  $\mathcal{H}_X$ .

**Theorem 1.6.**  $\mathcal{H}_{X, L}$  carries an irreducible unitary representation of  $\widetilde{A(X)}$ , with the center acting by the character corresponding to  $L$ .

*Proof.* Since the measure  $\mu_f^L$  is  $\widetilde{A(X)}$ -invariant,  $\mathcal{H}_X$  carries a representation of  $\widetilde{A(X)}$  which is unitary for the inner product defined by  $\mu_f^L$ . The statement about the action of the center follows immediately from the definitions. We need to show that the representation is irreducible. First assume  $X$  to be connected. Irreducibility then follows easily from the decomposition  $X \cong W \times Y$  as in the proof of Proposition 1.5: this reduces the statement to the Examples 1.9 i) and ii) below. When  $X$  is disconnected, it suffices to observe that  $\ell^2(\pi_0(X))$  carries the unique irreducible representation of the Heisenberg group constructed from  $\pi_0(X)$  and its Pontryagin dual.  $\square$

**Corollary 1.7.** The quantization  $\mathcal{H}_X$  is, up to isomorphism, independent of the specific complex structure in the polarization class.

*Proof.* This follows immediately from Theorem A.5, which states that, for a given polarization class,  $\widetilde{A(X)}$  has only one irreducible representation for a fixed character of the center.  $\square$

**Proposition 1.8.** For  $X$  connected, there is a canonical isomorphism

$$\text{Ind}_{\tilde{V}_X}^{\tilde{V}}(\mathcal{H}_X) \cong \mathcal{H}_V.$$

*Proof.* Let us first recall the construction of the induced representation in this case. Consider the inclusion  $V_X \hookrightarrow V$  and notice that there is an exact sequence

$$0 \rightarrow V_X \rightarrow V \rightarrow X^d \rightarrow 0,$$

where  $X^d \cong \text{Hom}(V_{x_0}, \mathbb{T})$  is a finite dimensional torus. Therefore we have

$$\text{Ind}_{\tilde{V}_X}^{\tilde{V}}(\mathcal{H}_X) := L^2(X^d, \mathcal{H}_X)$$

with the inner product defined with respect to the Haar measure on  $X^d$ . But since  $X^d$  is Pontryagin dual to  $V_{x_0}$ , the Hilbert space above is exactly the spectral decomposition of  $\mathcal{H}_V$  with respect to the abelian subgroup  $V_{x_0} \subset \tilde{V}$ : as a representation

of an abstract locally compact abelian group, there is a disintegration

$$\mathcal{H}_V \cong \int_{X^d}^{\oplus} \mathcal{H}_{X,\zeta} d\mu(\zeta),$$

cf. Proposition A.7. Elements in  $\tilde{V}$  that do not commute with  $V_{x_0}$  act by translations over  $X^d$  and because the resulting action is transitive, the spectral measure class must be  $X^d$ -invariant. Of course, the Haar measure is a unique representant of this class, and therefore the two representations of  $\tilde{V}$  are canonically isomorphic.  $\square$

Since the representation of  $\tilde{V}$  on  $\mathcal{H}_V$  is irreducible, this gives another proof of irreducibility of the representation of  $\tilde{V}_X$  on  $\mathcal{H}_X$  by Mackey's theorem. Of course, a similar result holds for non-connected  $X$ .

**Example 1.9.** Let us give some examples of this quantization:

- i) When  $X = V$  is a symplectic vector space, the quantization  $\mathcal{H}_V$  is simply the standard irreducible representation of the Heisenberg group  $\tilde{V}$  associated to the given polarization [BSZ]. Instead of the construction above with the Gaussian measure, one can construct the representation directly on the symmetric Hilbert space  $S(A)$ , where  $V_{\mathbb{C}} = A \oplus \bar{A}$  is the decomposition into  $+i$  and  $-i$  eigenspaces of  $J$ , cf. [S1].
- ii) In case of a finite dimensional polarized abelian variety  $(X, L)$ , the prescription above produces the Hilbert space  $H^0(X, L)$  equipped with the Liouville inner product. Of course, this is just the space of Theta-functions. In this case,  $\tilde{V}_X$  is a finite Heisenberg group. That  $H^0(X, L)$  is the unique irreducible representation of this group is well known, see e.g. [Mu2, §1.3]. The torus  $H^1(X, \mathbb{T})$  parameterising the inequivalent quantizations of  $X$  is in this context usually referred to as the *dual torus*  $X^d$ .

**1.3. The universal family.** Next, we will introduce, for each complex structure in the polarization class, a canonical rigging of the Hilbert space  $\mathcal{H}_X$ . Since this Hilbert space consists of square integrable sections of an extension of the line bundle  $L$  to  $X^*$ , one obtains, by restriction to  $X$ , a canonical map  $\mathcal{H}_X \rightarrow \hat{E}_X$ , where  $\hat{E}_X := \Gamma_{hol}(X, L)$ . This map is injective by the fact that the inclusion  $X \hookrightarrow X^*$  is dense, i.e., holomorphic sections of  $L$  on  $X^*$  are uniquely determined by their restriction to  $X$ . When equipped with the topology of uniform convergence on compact subsets,  $\hat{E}_X$  is a complete, locally convex topological vector space, and the map  $\mathcal{H}_X \hookrightarrow \hat{E}_X$  is a dense continuous inclusion. Let  $\check{E}_X$  be the continuous anti-dual of  $\hat{E}_X$ , i.e., the complex conjugate space of all linear continuous functionals on  $\hat{E}_X$ . Restricting such a functional to  $\mathcal{H}_X$  defines, by Riesz' theorem, a vector in  $\mathcal{H}_X$ . This defines a dense embedding of  $\check{E}_X$  into  $\mathcal{H}_X$  and gives a rigging of this Hilbert space;

$$\check{E}_X \hookrightarrow \mathcal{H}_X \hookrightarrow \hat{E}_X,$$

where each of the inclusions is dense and continuous. For  $q \in L \setminus \{0\}$ , evaluation at  $q$  defines an element  $ev_q \in \check{E}_X$ , because the sections are holomorphic. By the



inclusions above, this gives maps as in the following diagram:

$$\begin{array}{ccc} L \setminus \{0\} & \longrightarrow & \mathcal{H}_X \setminus \{0\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}\mathcal{H}_X \end{array}$$

Notice that the right hand side is just the universal holomorphic line bundle over the projective Hilbert space. The inner-product in  $\mathcal{H}_X$  induces a natural hermitian metric on this bundle whose curvature is given by the Fubini–Study form on  $\mathbb{P}(\mathcal{H}_X)$ . Now we have:

**Proposition 1.10.** *The map  $X \rightarrow \mathbb{P}(\mathcal{H}_X)$  is an embedding of Kähler manifolds, i.e., the pull-back of the Fubini–Study form on  $\mathbb{P}(\mathcal{H}_X)$  equals  $\omega$ . Consequently, the pull-back of the universal bundle to  $X$  is isomorphic, via the upper map in the diagram, to the prequantum line bundle  $L$ .*

*Proof.* It is easy to check that the map  $X \rightarrow \mathbb{P}(\mathcal{H}_X)$  is a holomorphic embedding. To see that it is in fact Kähler, consider  $\mathbb{P}(\mathcal{H}_X)$  as a Kähler manifold with symplectic form given by the Fubini–Study metric. As stated above, the universal bundle given by  $\mathcal{H}_X \setminus \{0\}$  gives a pre-quantization of  $\mathbb{P}(\mathcal{H}_X)$ . Because  $A(X)$ , by its embedding in  $PU(\mathcal{H}_X)$ , acts on  $\mathbb{P}(\mathcal{H}_X)$  by Kähler isometries, this prequantum line bundle determines a central extension of  $A(X)$  uniquely determined by the Fubini–Study form. But this central extension must be equal to the Heisenberg extension of  $A(X)$  defined by the symplectic form  $\omega$  on  $X$  since this is the group that actually acts by unitary transformations on  $\mathcal{H}_X$ .

Next consider the pull-back of the universal bundle to  $X$ . By the argument above, its curvature equals  $\omega$ . Since the representation of  $\widetilde{A(X)}$  on  $\mathcal{H}_X$  corresponds to the same character of  $V_{x_0}$  as the one that determines the line bundle  $L$  over  $X$ , cf. Proposition 1.6, the two line bundles are isomorphic.  $\square$

Restricting attention to the base point  $x_0 \in X$ , we therefore have, for each complex structure, a distinguished ray in the Hilbert space  $\mathcal{H}_X$ . This yields the following point of view on quantization: a quantization of a polarized affine symplectic manifold is an irreducible representation of the associated Heisenberg group (cf. Prop. 1.2) with the additional data of a ray in the Hilbert space for each complex structure in the given polarization class. As such, these rays, for varying complex structure form a line bundle over the Siegel upper half space  $\mathcal{J}(V)$ , cf. §A.2.1.

**Theorem 1.11.** *Let  $X$  be a polarized affine symplectic manifold modelled on a symplectic vector space  $V$ . The tensor product*

$$\text{Det}_V^{-1/2} \otimes \mathcal{H}_X$$

*carries a canonical inner product and forms, for varying complex structure, a flat holomorphic Hilbert bundle over  $\mathcal{J}(V)$*

*Proof.* Let  $L_X$  be the line bundle given by the ray spanned by the basepoint  $x_0 \in X$  as described above. Given the fact that  $\mathcal{H}_X$  is an irreducible representation of  $\tilde{V}_X$ , exactly the same reasoning as in the end of §A.2.1 leads to the conclusion that the tensor product  $L_X^* \otimes \mathcal{H}_X$  is canonically flat as a bundle of Hilbert spaces. But it follows from Proposition 1.8 that  $L_X$  is canonically isomorphic to  $\text{Det}_V^{1/2}$ .  $\square$

**Remark 1.12.** When  $X$  is compact, i.e., an abelian variety, the quantization is finite dimensional and the theorem above implies that the connection associated to the holomorphic hermitian vector bundle  $\mathcal{H}_X$  is projectively flat. This is the connection described in [W] leading to the heat equation satisfied by  $\Theta$ -functions.

Let  $\text{Aut}(X) \subseteq \text{Sp}(V)$  be defined as

$$\text{Aut}(X) := \{\text{Symplectic automorphisms of } (V_{x_0}, \omega)\}.$$

When  $X$ , and therefore  $V$ , is polarized, define  $\text{Aut}_{\text{res}}(X) = \text{Aut}(X) \cap \text{Sp}_{\text{res}}(V)$ , where  $\text{Sp}_{\text{res}}(V)$  is the restricted symplectic group, cf. §A.2.1. Since elements of  $\text{Aut}(X)$  need not preserve the complex structure, this group a priori does not act on the quantization  $\mathcal{H}_X$ . However, elements of  $\text{Aut}_{\text{res}}(X)$  map one point of  $\mathcal{J}(V)$  to another and therefore Proposition 1.11 gives:

**Corollary 1.13.**  $\mathcal{H}_X$  carries a projective unitary representation of  $\text{Aut}_{\text{res}}(X)$ . It extends the representation of  $\widetilde{A(X)}$  to the semi-direct product  $\widetilde{\text{Aut}_{\text{res}}(X)} \ltimes \widetilde{A(X)}$

**1.4. Symplectic reduction.** In this section we will study the reduction theory of an affine symplectic manifold with respect to (part of) its affine symmetries. Let  $X$  be an affine symplectic manifold modelled on  $V$ , which is assumed to be connected. By definition, the vector space  $V$  acts symplectically on  $X$ . Restricted to  $V_X$ , this action is even Hamiltonian with moment map  $J : X \rightarrow \text{Lie}(V_X^*)$  given by

$$(1.4) \quad J(x) = -\omega(v_x, -)|_{\text{Lie}(V_0^*)},$$

where  $v_x \in V$  is a pre-image of  $x$  under  $\pi$ , cf. the exact sequence (1.1). This may be seen to be independent of the choice of the lift, i.e., up to the action of  $V_{x_0}$ , by realizing that  $V_X \subseteq V$  is the projection of the commutant of  $\tilde{V}_{x_0}$  in  $\tilde{V}$ . Notice that the moment map is only *affine*-equivariant, with respect to the cocycle defined by the symplectic form  $\omega$ . This is exactly the cocycle that defines the Heisenberg extension of  $V_X \subseteq V$ . A closed abelian subgroup  $B \subseteq A(X)$  is said to be isotropic if the induced central extension  $\tilde{B}$  is abelian. As such, the extension must be trivial, but not canonically, i.e., depends on the choice of a splitting  $\chi : B \rightarrow \mathbb{T}$  satisfying

$$\chi(b_1 b_2) = \chi(b_1) \chi(b_2) e^{i\pi \omega(b_1, b_2)}.$$

Additively, this is given by a map  $\psi : B \rightarrow \mathbb{R}$  satisfying  $\psi(b_1 + b_2) = \psi(b_1) + \psi(b_2) + \omega(b_1, b_2)$ . Therefore,  $J_B := J + \psi$  defines a moment map which is  $B$ -equivariant and one can consider the symplectically reduced space

$$X_{\text{red}} = X // B := J_B^{-1}(0) / B.$$

**Proposition 1.14.** Let  $(B, \chi)$  be an isotropic subgroup of  $A(X)$  with a choice of splitting. When  $B$  acts freely on  $X$ ,  $X_{\text{red}}$  is an affine symplectic manifold modelled on

$$\text{Lie}(B)^\circ / \text{Lie}(B),$$

where  $\text{Lie}(B)^\circ$  denotes the symplectic complement of  $\text{Lie}(B) \subseteq V$  with respect to  $\omega$ . A polarization of  $X$  induces a polarization of  $X_{\text{red}}$ , and a prequantum line bundle  $L$  on  $X$  induces a prequantum line bundle  $L_\chi$  on  $X_{\text{red}}$ .

*Proof.* First notice that since  $B$  is assumed to act freely, one has in fact  $B \subset V_X$ , and  $B$  fits into a short exact sequence of abelian groups

$$0 \rightarrow \text{Lie}(B) \rightarrow B \rightarrow \pi_0(B) \rightarrow 0.$$

It is easy to see that the splitting  $\psi$  must be zero on  $\text{Lie}(B)$ , i.e.,  $\text{Lie}(B) \subset V$  is an isotropic subspace in the sense that  $\omega|_{\text{Lie}(B)} = 0$ . Consider the zero locus of the moment map,  $J_B^{-1}(0) \subseteq X$ . It follows at once from the formula (1.4) for the moment map that the symplectic complement  $\text{Lie}(B)^\circ$  is the maximal subspace of  $V$  acting on  $J_B^{-1}(0)$ . Since the action of  $V$  on  $X$  is assumed to be transitive on each connected component, the same will be true for the induced action of  $\text{Lie}(B)^\circ / \text{Lie}(B)$  on  $X_{\text{red}}$ . Since  $\pi_1(X)$  and  $\pi_0(B)$  are assumed to be finitely generated lattices, the isotropy groups of this action will be finitely generated as well. This proves that  $X \parallel B$  is an affine symplectic manifold modelled on  $\text{Lie}(B)^\circ / \text{Lie}(B)$  in the sense of Definition 1.1.

Let  $L$  be a prequantum line bundle over  $X$ . It is easy to check that

$$L_X := L|_{J_B^{-1}(0)} \times_{\tilde{B}} \mathbb{C}_X$$

defines a prequantum line bundle over  $X \parallel B$ . Finally, the fact that a polarization  $J$  of  $V$  induces a polarization of  $\text{Lie}(B)^\circ / \text{Lie}(B)$ , follows from the canonical isomorphism  $V \parallel \text{Lie}(B) \cong V / \text{Lie}(B)_\mathbb{C}$ ; the right hand side has a canonical complex structure compatible with the symplectic form.  $\square$

**Remark 1.15.** Notice that the first two homotopy groups of  $X_{\text{red}}$  fit into an exact sequence of abelian groups

$$(1.5) \quad 0 \rightarrow \pi_1(X) \rightarrow \pi_1(X_{\text{red}}) \rightarrow \pi_0(B) \rightarrow \pi_0(X) \rightarrow \pi_0(X_{\text{red}}) \rightarrow 0.$$

Of course, the higher homotopy groups are trivial.

**Proposition 1.16.** *The Heisenberg groups associated to affine symplectic manifolds are related under reduction by*

$$\tilde{V}_{X_{\text{red}}} / \pi_1(\widetilde{X_{\text{red}}}) \cong \tilde{B}^\perp / B,$$

where the commutant  $\tilde{B}^\perp$  is taken in the Heisenberg group  $\widetilde{A(X)} / \pi_1(\widetilde{X})$

*Proof.* The proof is straightforward, we omit the details.  $\square$

**1.5. Induced representations.** In this section we will define a quantized notion of reduction for affine symplectic manifolds, denoted  $\mathcal{QR}$ , and prove that “quantization commutes with reduction”, i.e., that there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{Q}} & \mathcal{H}_X \\ \text{Symplectic Reduction} \downarrow & & \downarrow \text{Quantum Reduction} \\ X_{\text{red}} & \xrightarrow{\mathcal{Q}} & \mathcal{H}_{X_{\text{red}}} \end{array}$$

In this diagram  $\mathcal{SR}$  denotes the operation of symplectic reduction with respect to an isotropic subgroup, and  $\mathcal{Q}$  is the quantization functor as defined in §1.2. This means that we assume  $X$  to be polarized, and  $X_{\text{red}}$  to carry the induced polarization class as in Proposition 1.14. Then we can consider quantization to give a flat Hilbert bundle over  $\mathcal{J}(V)$ , the Siegel upper half space of  $V$ , the symplectic vector space modelling  $X$ .

Our approach may be motivated by the classical example of Theta-functions. Let us start by the following remark: suppose that  $\mathcal{H}$  is a unitary representation

of a group  $G$  which is discretely reducible, with  $\mathcal{H}_i$ ,  $i \in I$  being the irreducible representations that can occur in its decomposition. Then the space  $\text{Hom}_G(\mathcal{H}_i; \mathcal{H})$  of intertwiners carries a canonical inner product given by

$$\langle \psi_1, \psi_2 \rangle := \psi_1^* \psi_2 \in \text{Hom}_G(\mathcal{H}_i; \mathcal{H}_i) = \mathbb{C},$$

for  $\psi_1, \psi_2 \in \text{Hom}_G(\mathcal{H}_i; \mathcal{H})$ . With this inner product, the canonical map

$$\bigoplus_{i \in I} \text{Hom}_G(\mathcal{H}_i; \mathcal{H}) \otimes \mathcal{H}_i \xrightarrow{\cong} \mathcal{H}$$

defined by  $\psi_i \otimes v_i \mapsto \psi_i(v_i) \in \mathcal{H}$ , for  $\psi_i \in \text{Hom}_G(\mathcal{H}_i; \mathcal{H})$  and  $v_i \in \mathcal{H}_i$ , is a unitary isomorphism providing the decomposition of  $\mathcal{H}$  into irreducibles.

**Theorem 1.17.** *Let  $(X, L_\chi)$  be a polarized abelian variety with  $X = V/\Lambda$  where  $\Lambda$  is a full isotropic lattice in a finite dimensional complex symplectic vector space  $(V, \omega)$ , with a choice of splitting  $\chi$ . Then the associated space of Theta-functions is given by*

$$H^0(X, L_\chi) \cong \text{Hom}_{\tilde{V}} \left( \text{Ind}_{\tilde{\Lambda}}^{\tilde{V}}(\mathbb{C}_\chi), \mathcal{H}_V \right).$$

*Proof.* This theorem can be proved at once using the material of the appendix, but in order to explicate the simplicity of the argument, we will give a proof from scratch. By Proposition 1.6, the left hand side carries an irreducible representation of the Heisenberg group  $\tilde{\Lambda}^\circ$ , with the center  $\tilde{\Lambda}$  acting via  $\chi$ . Choose a Lagrangian lattice  $L$  intermediate between  $\Lambda$  and  $\Lambda^\circ$  together with an extension  $\chi_L$  of  $\chi$ ; we have  $\Lambda \subseteq L \subseteq \Lambda^\circ$ , and  $L$  is isotropic and maximal. Functoriality of induction then gives a natural isomorphism

$$\text{Ind}_{\tilde{\Lambda}}^{\tilde{V}} \cong \text{Ind}_L^{\tilde{V}} \circ \text{Ind}_{\tilde{\Lambda}}^{\tilde{L}}.$$

Applied to the representation  $\mathbb{C}_\chi$ , we first compute

$$\text{Ind}_{\tilde{\Lambda}}^{\tilde{L}}(\mathbb{C}_\chi) = L^2(L/\Lambda; L_\chi),$$

but this representation of  $\tilde{L}$  has a canonical extension to  $\tilde{\Lambda}_{op}^\circ$ , viz. the “Schrödinger representation” associated to  $L$ . This is the irreducible representation of the Heisenberg extension  $\tilde{\Lambda}_{op}^\circ$  with the center acting via  $\chi$ , i.e., precisely isomorphic to  $\mathcal{H}_{X,\chi}^*$ . With this we compute

$$\begin{aligned} \text{Ind}_{\tilde{\Lambda}}^{\tilde{V}}(\mathbb{C}_\chi) &\cong \text{Ind}_L^{\tilde{V}} \left( \mathcal{H}_{X,\chi}^* \right) \\ &= L^2(V/L; \tilde{V} \times_{\tilde{\Lambda}} \mathcal{H}_{X,\chi}^*) \\ &\cong L^2(V/L; L_{\chi_L}) \otimes \mathcal{H}_{X,\chi'}^* \end{aligned}$$

where in the last line we have used the splitting  $\chi_L$  to extract  $\mathcal{H}_{X,\chi}^*$  from the fiber. Now, the first factor in the tensor product is simply the representation of  $\tilde{V}$  induced from  $\mathbb{C}_{\chi_L}$ , which is irreducible by Mackey’s theorem and therefore killed by taking  $\text{Hom}_{\tilde{V}}(-, \mathcal{H}_{\tilde{V}})$ . The result now follows.  $\square$

**Remark 1.18.** The subspace  $\tilde{\Lambda}$ -invariants in  $\mathcal{H}_V$  is trivial, i.e., equal to  $\{0\}$ . This can be seen as follows: by a similar line of reasoning, this time using the chain of

inclusions  $L \subseteq \Lambda^\circ \subset V$ , one finds

$$\begin{aligned} \mathcal{H}_V &\cong \text{Ind}_{\tilde{\Lambda}^\circ}^{\tilde{V}} \circ \text{Ind}_{\tilde{L}}^{\tilde{\Lambda}^\circ}(\mathbb{C}_\chi) \\ &\cong \text{Ind}_{\tilde{\Lambda}^\circ}^{\tilde{V}}(\mathcal{H}_{\tilde{\Lambda}^\circ, \chi}) \\ &\cong L^2(X^d, \mathcal{H}_{X, \chi}). \end{aligned}$$

This is, of course, nothing but the spectral decomposition of  $\mathcal{H}_V$  under the representation of the abelian subgroup  $\Lambda$ , acting via the splitting  $\chi$ . It clearly shows that the “reduced space”  $\mathcal{H}_X$  is not a closed subspace of invariants of any kind of  $\mathcal{H}_V$ . In the literature, cf. e.g. [Mu2], this is usually circumvented by taking invariants under  $\Lambda$  in a certain distributional completion of  $\mathcal{H}_V$ . The main advantage of the point of view expressed in the theorem is its manifest unitarity: it only refers to the Hilbert spaces involved in the quantization. As such, one easily shows that the isomorphism is completely natural, i.e., as projectively flat Hilbert bundles over  $\mathcal{J}(V)$ . This point of view on Theta-functions, i.e., as intertwiners between certain induced representations, originated in [Ma].

To proceed in this way for general (infinite dimensional) affine symplectic manifolds, one needs an infinite dimensional version of induced representations. This is developed in Appendix A.4. With this the general case presents no difficulty:

**Theorem 1.19.** *Let  $(X, L)$  be a polarized affine symplectic manifold. Let  $(B, \chi)$  be an isotropic subgroup of  $A(X)$  with a choice of splitting, which acts freely on  $X$ . There is an isomorphism of projectively flat Hilbert bundles*

$$\mathcal{H}_{X//A, L_\chi} \cong \text{Hom}_{\widetilde{A(X)}} \left( \text{Ind}_{\tilde{V}_{x_0} \times \tilde{B}}^{\widetilde{A(X)}}(L_{x_0} \otimes \mathbb{C}_\chi), \mathcal{H}_{X, L} \right).$$

*Proof.* First notice that a polarization of  $X$  determines a unique polarization class of  $A(X)$ : this is immediate from Definition A.4. Therefore the induced representation is well-defined as in Definition A.8. By assumption,  $B \cap Z(A(X)) = B \cap V_{x_0} = \{e\}$ , and since we also induce from the character of  $\tilde{V}_{x_0}$ , Theorem A.11 implies that the induced representation carries an irreducible positive energy representation of the generalized Heisenberg group

$$\widetilde{A(X)} \times (\tilde{B} \times \tilde{V}_{x_0})_{op}^\perp = \widetilde{A(X)} \times \tilde{B}_{op}^\perp.$$

By proposition 1.16, the right hand side is exactly the generalized Heisenberg group associated to  $X//B$ . Therefore, there is a canonical isomorphism

$$(1.6) \quad \text{Ind}_{\tilde{B} \times \tilde{V}_{x_0}}^{\widetilde{A(X)}}(L_{x_0} \otimes \mathbb{C}_\chi) = \mathcal{H}_{X, L} \otimes \mathcal{H}_{X//B, L_\chi}^*.$$

The result now follows immediately.  $\square$

The application of this theorem below amounts to the following special case: Let  $X_1$  and  $X_2$  be two affine symplectic manifolds modelled on  $(V, \omega)$ , and let  $B \subset V$  be any closed abelian subgroup acting freely on  $X_1$  and  $X_2$ , i.e., not necessarily isotropic. Then  $B$  can be considered as an isotropic subgroup of  $V \times \tilde{V}$  by the diagonal embedding, where  $\tilde{V}$  denotes  $V$  with the symplectic form  $-\omega$ . The

induced extension is even canonically split. It therefore follows that

$$\mathcal{H}_{(X_1 \times \bar{X}_2) // B} \cong \text{Hom}_{\widetilde{A(X_1)} \times \widetilde{A(\bar{X}_2)}_{op}} \left( \text{Ind}_B^{\widetilde{A(X_1)} \times \widetilde{A(\bar{X}_2)}_{op}} (\mathbb{C}), \mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}^* \right).$$

## 2. QUANTIZATION OF THE MODULI SPACE OF HOLOMORPHIC TORUS-BUNDLES

**2.1. Notation.** Let us briefly introduce some notation in relation to abelian gauge theory. Let  $T$  be a compact abelian Lie group with Lie algebra  $\mathfrak{t}$ . For any compact manifold  $M$ , possibly with boundary, let  $\mathcal{A}(M)$  be the space of smooth connections on the trivial principal  $T$ -bundle over  $M$ . This is an infinite dimensional affine Fréchet space modelled on  $\Omega^1(M, \mathfrak{t})$ , the space of 1-forms with values in  $\mathfrak{t}$ . The gauge group  $T(M) := C^\infty(M, T)$  is defined to be the Fréchet Lie group of smooth maps from  $M$  into  $T$ , with Lie algebra given by  $\mathfrak{t}(M) := \Omega^0(M, \mathfrak{t})$ . Writing  $T = \mathfrak{t}/\Lambda$ , where  $\Lambda := \text{Hom}(\mathbb{T}, T)$  is the integral lattice, we have a short exact sequence

$$0 \rightarrow \underline{\Lambda} \rightarrow \underline{\mathfrak{t}} \rightarrow \underline{T} \rightarrow 0,$$

of sheaves of smooth functions. The long exact sequence in cohomology then gives

$$0 \rightarrow H^0(M, \Lambda) \rightarrow \mathfrak{t}(M) \rightarrow T(M) \rightarrow H^1(M, \Lambda) \rightarrow 0,$$

from which we read off that

$$\pi_0(T(M)) = H^1(M, \Lambda), \quad \pi_1(T(M)) = H^0(M, \Lambda).$$

The gauge group  $T(M)$  naturally acts on  $\mathcal{A}(M)$  by the affine transformations

$$(2.1) \quad \varphi \cdot A = A - d\varphi\varphi^{-1},$$

where  $A \in \mathcal{A}(M)$  and  $\varphi \in T(M)$ . This action is generated by the fundamental vector fields  $-d\zeta$ , where  $\zeta \in \Omega^0(M, \mathfrak{t})$ . Below, we will use these objects only in  $\dim(M) = 1$  and 2.

**2.2. Positive energy representations of  $LT$ .** In this section we will discuss the notion of a positive energy representation of the gauge group  $T(S)$  associated to a compact oriented 1-manifold  $S$ . When  $S$  is connected, it is of course diffeomorphic to  $S^1$ , and the group  $T(S^1)$ , called the loop group of  $T$ , is denoted by  $LT$ . Let  $\text{Rot}(S^1)$  be the group of rotations of the circle, i.e., the group  $S^1$  acting on itself. Recall the definition of a positive energy representation:

**Definition 2.1.** A positive energy representation of  $LT$  is a projective unitary representation on a Hilbert space  $\mathcal{H}$ , given by a strongly continuous homomorphism  $\pi : LT \rightarrow PU(\mathcal{H})$ , which has an extension to the semi-direct product  $\text{Rot}(S^1) \ltimes LT$  such that the action of  $\text{Rot}(S^1) \cong \mathbb{T}$  can be lifted to an action by non-negative characters.

By Stone's theorem the action of  $\text{Rot}(S^1)$  is generated by a selfadjoint operator on  $\mathcal{H}$  and the positive energy condition requires this operator to have positive spectrum. As a result of this condition, any positive energy representation is discretely reducible. The irreducible representations can be constructed and classified as in [PS, §9.5], or below. A striking consequence of the positive energy condition is that the action of  $\text{Rot}(S^1)$  in fact extends to  $\text{Diff}^+(S^1)$  resulting in a projective unitary representation of the semi-direct product  $\text{Diff}^+(S^1) \ltimes LT$ . This allows us

to extend the notion of a positive energy representation to all gauge groups  $T(S)$  for  $S$  a compact oriented 1-manifold.

By the positive energy condition, the projective representations above are actually true representations of a *smooth* central extension

$$(2.2) \quad 1 \rightarrow \mathbb{T} \rightarrow \widetilde{LT} \rightarrow LT \rightarrow 1.$$

These extensions are classified as follows: since  $LT$  is abelian, any such extension is topologically trivial and, up to isomorphism, determined by the commutator map  $s : LT \times LT \rightarrow \mathbb{T}$ , cf. §A.1. Taking the fundamental group, this defines a quadratic form  $q : \Lambda \times \Lambda \rightarrow \mathbb{Z}$  on  $\pi_1(T)$ , which classifies the extension up to isomorphism.

We will now assume, cf. Remark 2.2, that the quadratic form  $q$  turns the lattice into an even one. Then the corresponding central extension is constructed as follows: extend  $q$  to an inner product  $\langle \cdot, \cdot \rangle : \mathfrak{t} \times \mathfrak{t} \rightarrow \mathbb{R}$  on the Lie algebra  $\mathfrak{t}$ . We also choose an integral bilinear form  $B$  on  $\Lambda$  such that

$$(2.3) \quad B(\lambda_1, \lambda_2) + B(\lambda_2, \lambda_1) = \langle \lambda_1, \lambda_2 \rangle,$$

for all  $\lambda_1, \lambda_2 \in \Lambda$ . This is possible because the lattice is even, and necessary to construct an explicit central extension: the isomorphism class however does not depend on the choice of  $B$ .

The metric determines a cocycle  $\omega : Lt \times Lt \rightarrow \mathbb{R}$  on the Lie algebra  $Lt$  by

$$(2.4) \quad \omega(\xi, \eta) := \int_S \langle \xi, d\eta \rangle.$$

Furthermore, there is a decomposition

$$(2.5) \quad LT \cong \Lambda \times V(S) \times T,$$

where  $V(S) = Lt/\mathfrak{t}$ . The cocycle (2.4) turns  $V(S)$  into a symplectic vector space, which therefore has a unique Heisenberg extension. The bilinear form  $B$  defines an embedding  $\Lambda \rightarrow \hat{T}$ , the Pontryagin dual of the compact torus, and this defines a cocycle as in (1.2). Putting these two extensions together defines the central extension of  $LT$ : it clearly is a generalized Heisenberg group as defined in §A.1 with center given by  $A \times \mathbb{T}$ , where  $A$  is the finite group  $A := \Lambda^\circ/\Lambda$  and

$$\Lambda^\circ = \{\mu \in \mathfrak{t}, \langle \mu, \lambda \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda\} \cong \text{Hom}(T, \mathbb{T})$$

is the dual lattice. Notice that the nondegenerate bilinear form induced by the metric defines a Pontryagin self-duality  $A \cong \hat{A}$ .

There is a canonical polarization  $V(S^1)_\mathbb{C} = V_+(S^1) \oplus V_-(S^1)$  by the decomposition into positive and negative Fourier modes, or, equivalently, the Hilbert transform. This defines a polarization class on  $LT$  in the sense of Definition A.4 which is invariant under the action of  $\text{Diff}^+(S^1)$ . As shown in [PS, §9.5], it is exactly this class that leads to positive energy representations in the sense of Definition 2.1. By Theorem A.5, the irreducible representations at level  $q$  now correspond to the characters  $\varphi \in \hat{A}$ . To give an explicit construction, consider the unique Heisenberg representation  $\mathcal{H}_{V(S^1)}$  of the Heisenberg extension of  $V(S^1)$  defined by the polarization. Choose a lift  $\lambda \in \Lambda^\circ$ , and let  $T$  act on  $\mathcal{H}_{V(S^1)}$  via  $\lambda$ . This yields an

irreducible representation  $\mathcal{H}_\lambda$  of the unit component  $LT_0$ , and with this we define

$$\begin{aligned}\mathcal{H}_\varphi &:= \text{Ind}_{LT_0}^{\widetilde{LT}}(\mathcal{H}_\lambda) \\ &= \bigoplus_{\mu \in \Lambda} \mathcal{H}_{\lambda+\mu}.\end{aligned}$$

This clearly defines an irreducible representation of  $\widetilde{LT}$  which only depends on  $\varphi$ , the image of  $\lambda$  in  $\hat{A}$ . Notice that only when the level turns  $\Lambda$  into a unimodular lattice, the central extension of  $LT$  is a Heisenberg group, i.e., has a unique irreducible representation.

**Remark 2.2.** In the above we have assumed that the level, i.e., the quadratic form  $q$  on  $\pi_1(T)$ , turns  $\Lambda$  into an *even* lattice. This condition has a topological origin: it is actually better to view the quadratic form as a class  $q \in H^1(T; \mathbb{Z}) \otimes H^1(T; \mathbb{Z})$ . Since  $H^1(T; \mathbb{Z}) = H^2(BT; \mathbb{Z})$ , and  $BLT \simeq T \times BT$ , we have  $q \in H^3(BLT; \mathbb{Z})$ . It is this cohomology class that classifies, also for nonabelian compact Lie groups, the central extensions that arise from positive energy representations.

Because of the homotopy equivalence  $BLT \simeq LBT$ , there is a transgression map

$$H^4(BT; \mathbb{Z}) \rightarrow H^3(BLT; \mathbb{Z}).$$

Because the cohomology  $H^*(BT; \mathbb{Z})$  is generated by the first Chern class  $c_1 \in H^2(BT; \mathbb{Z})$ , the group  $H^4(BT; \mathbb{Z})$  equals the space of integral even bilinear forms on  $\pi_1(T)$ . We therefore see that our condition on the quadratic form  $q$  means that the corresponding class in  $H^3(BLT; \mathbb{Z})$  is transgressed from  $H^4(BT; \mathbb{Z})$ . For example, for  $T = \mathbb{T}$ , both cohomology groups are isomorphic to  $\mathbb{Z}$ , but the map turns out to be multiplication by 2.

The Conformal Field Theory (CFT) that we are about to describe comes from a Topological Quantum Field Theory (TQFT) in dimension 3, called abelian Chern–Simons theory, classified by  $H^4(BT; \mathbb{Z})$ . From the point of view of loop groups, the structure defined by this theory, e.g., the modular tensor structure on the representation category, is therefore only defined for even levels, i.e., corresponding to inner products for which the lattice  $\Lambda$  is even. To treat the odd level, corresponding to half-integer Chern–Simons theory, one needs to refine to a *spin*-TQFT, in which manifolds are assumed to be equipped with a spin-structure. This TQFT is described in [BM]. In this paper we will mainly deal with the case of an even level, but will remark where the spin structure comes in when treating the odd-level case. With this, the construction of the corresponding spin-CFT presents no difficulty.

**2.3. The moduli space of holomorphic  $T_{\mathbb{C}}$ -bundles.** In this section we will describe how the theory of affine symplectic manifolds can be used to quantize certain moduli spaces  $T$ -bundles over surfaces with boundaries. Before turning to quantization, let us first describe this moduli space in some more detail, cf. [AB, Do, MW]. Let  $\Sigma$  be a two dimensional smooth oriented surface, possibly with smooth boundaries. Consider  $\mathcal{A}(\Sigma)$ , the space of connections on the trivial  $T$ -bundle over  $\Sigma$ . In two dimensions, this space carries a symplectic form

$$(2.6) \quad \omega(\alpha, \beta) = \int_{\Sigma} \langle \alpha, \beta \rangle,$$



for  $\alpha, \beta \in T_A \mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{t})$ , so that the pair  $(\mathcal{A}(\Sigma), \omega)$  is an affine symplectic manifold. The action of the gauge group  $T(\Sigma)$  on  $\mathcal{A}(\Sigma)$ , cf. equation (2.1), preserves this symplectic form and is Hamiltonian with associated moment map given by

$$\langle \mu(A), \xi \rangle = \int_{\Sigma} \langle F_A, \xi \rangle - \int_{\partial \Sigma} i_{\partial \Sigma}^* \langle A, \xi \rangle,$$

for  $\xi \in \Omega^0(\Sigma, \mathfrak{t})$ , where  $i_{\partial \Sigma} : \partial \Sigma \hookrightarrow \Sigma$  is the canonical inclusion and  $F_A = dA$  denotes the curvature of a connection. Let  $T_{\partial}(\Sigma)$  be the subgroup of  $T(\Sigma)$  consisting of gauge transformations which are trivial at the boundary. This gauge group fits into an exact sequence of the form

$$(2.7) \quad 1 \rightarrow T_{\partial}(\Sigma) \rightarrow T(\Sigma) \rightarrow T(\partial \Sigma) \rightarrow H^2(\Sigma, \partial \Sigma; \Lambda) \rightarrow 0.$$

Pick a  $\chi \in \Omega_{\Lambda}^2(\Sigma, \mathfrak{t})$ , i.e., a  $\mathfrak{t}$ -valued 2-form for which  $\langle [\Sigma], \chi \rangle \in \Lambda$ , and define  $\mathcal{A}_{\chi}(\Sigma) \subset \mathcal{A}(\Sigma)$  to be the space of connections with curvature equal to  $\chi$ . With this, define the connected components of the moduli space as the quotient

$$(2.8) \quad \mathcal{M}_T^{[\chi]}(\Sigma) := \mathcal{A}_{\chi}(\Sigma) / T_{\partial}(\Sigma).$$

As a symplectic quotient, this space carries a canonical induced symplectic form. Notice that for two  $\chi, \chi' \in \Omega_{\Lambda}^2(\Sigma, \mathfrak{t})$  with  $\chi - \chi' = d\gamma$ , with  $\gamma \in \Omega^1(\Sigma)$  and  $\gamma|_{\partial \Sigma} = 0$ , translation by  $\gamma$  on  $\mathcal{A}(\Sigma)$  induces a canonical isomorphism  $\mathcal{M}_T^{[\chi]}(\Sigma) \cong \mathcal{M}_T^{[\chi']}(\Sigma)$  and therefore the construction only depends on the de Rham cohomology class  $[\chi] \in H^2(\Sigma, \partial \Sigma; \Lambda) \cong H_0(\Sigma; \Lambda)$  by Poincaré duality. Therefore, the total moduli space,

$$\mathcal{M}_T(\Sigma) := \coprod_{[\chi] \in H_0(\Sigma; \Lambda)} \mathcal{M}_T^{[\chi]}(\Sigma),$$

carries a canonical action of  $T(\partial \Sigma)$ : the kernel of the map  $T(\partial \Sigma) \rightarrow H^2(\Sigma, \partial \Sigma; \Lambda)$  above acts via its induced action from  $T(\Sigma)$  on  $\mathcal{A}(\Sigma)$ . The group of components  $\pi_0(T(\partial \Sigma)) = H^1(\partial \Sigma; \Lambda)$  acts by shifting the components according to the morphism  $H^1(\partial \Sigma; \Lambda) \rightarrow H^2(\Sigma, \partial \Sigma; \Lambda)$ . The resulting action of  $T(\partial \Sigma)$  clearly preserves the symplectic form and is Hamiltonian with moment map given by

$$(2.9) \quad [A] \mapsto -A|_{\partial \Sigma},$$

for  $[A] \in \mathcal{M}_T(\Sigma)$ .

**Example 2.3.** Let us consider some examples of these moduli spaces:

- i) When  $\partial \Sigma = \emptyset$ , the moduli space  $\mathcal{M}_T(\Sigma)$  is a disjoint union of finite dimensional symplectic tori  $[AB]$ ;

$$\mathcal{M}_T(\Sigma) \cong \text{Hom}(\pi_1(\Sigma), T) \times H^2(\Sigma; \Lambda).$$

For  $T = \mathbb{T}$ , the one-dimensional unitary group, this is nothing but the Jacobian of  $\Sigma$  consisting of isomorphism classes of line bundles.

- ii) For  $\Sigma = D$ , a disk, one finds  $\mathcal{M}_T(D) \cong LT/T$ , the affine coadjoint orbit of  $LT$  through zero.
- iii) For an annulus  $C$ , one finds  $\mathcal{M}_T(C) \cong T_{\omega}^*(LT)$ , where  $T_{\omega}^*$  means the twisted cotangent bundle: In this case,  $T_{\omega}^*(LT) = LT \times \Omega^1(S^1, \mathfrak{t})$ , viewed as a dense sub-bundle of the cotangent bundle with the canonical symplectic form twisted by the cocycle (2.4).

**Lemma 2.4.** *The moduli space and the gauge group fit into an exact sequence of groups*

$$1 \rightarrow H^0(\Sigma, T) \rightarrow T(\partial\Sigma) \rightarrow \mathcal{M}_T(\Sigma) \rightarrow H^1(\Sigma, T) \rightarrow 1.$$

*Proof.* The proof of this lemma is easy once one realizes that  $H^1(\Sigma, T)$  is the moduli space of flat  $T$ -bundles on  $\Sigma$  and the map  $\mathcal{M}_T(\Sigma) \rightarrow H^1(\Sigma, T)$  forgets the boundary framing.  $\square$

**Proposition 2.5.** *The moduli space  $\mathcal{M}_T(\Sigma)$  is an affine symplectic manifold. A complex structure defines a polarization whose class is independent of the specific choice of a point in the contractible space of complex structures on  $\Sigma$ .*

*Proof.* Let us first prove that  $\mathcal{M}_T(\Sigma)$  is an affine manifold. Consider the connected component  $\mathcal{M}_T^0(\Sigma)$  of the identity corresponding to the trivial bundle. Let

$$(2.10) \quad V(\Sigma) := \left\{ \alpha \in \Omega^1(\Sigma, \mathfrak{t}), d\alpha = 0 \right\} / d \left\{ \beta \in \Omega^0(\Sigma, \mathfrak{t}), \beta|_{\partial\Sigma} = 0 \right\}.$$

This vector space carries a symplectic form given by the formula (2.6), and  $V(\Sigma)$  acts on  $\mathcal{M}_T^0(\Sigma)$ , induced from the affine action  $A \mapsto A + \alpha$  of  $\Omega^1(\Sigma, \mathfrak{t})$  on  $\mathcal{A}(\Sigma)$ . Notice that the gauge group  $T_\partial(\Sigma)$  acts on  $\mathcal{A}(\Sigma)$  via the embedding  $T_\partial(\Sigma) \hookrightarrow \Omega^1(\Sigma, \mathfrak{t})$  given by  $\varphi \mapsto -d\varphi\varphi^{-1}$ . Since the Lie algebra of  $T_\partial(\Sigma)$  is exactly given by

$$\{\xi \in \Omega^0(\Sigma, \mathfrak{t}), \xi|_{\partial\Sigma} = 0\},$$

it follows from the definition (2.8) of  $\mathcal{M}_T^0(\Sigma)$  that the isotropy groups  $V(\Sigma)_{[A]}$ , for  $[A] \in \mathcal{M}_T^0(\Sigma)$ , of the action of  $V(\Sigma)$  on  $\mathcal{M}_T(\Sigma)_0$  are given by

$$V(\Sigma)_{[A]} = \pi_0(T_\partial(\Sigma)) = H^1(\Sigma, \partial\Sigma; \Lambda).$$

This is a finitely generated lattice, proving that the moduli space  $\mathcal{M}_T(\Sigma)$  is affine symplectic in the sense of Definition 1.1.

A complex structure on  $\Sigma$  turns  $\mathcal{A}(\Sigma)$  into a complex Kähler manifold by the Hodge  $*$ -operator  $*$  :  $\Omega^1(\Sigma, \mathfrak{t}) \rightarrow \Omega^1(\Sigma, \mathfrak{t})$  satisfying  $*^2 = -1$ , with associated Kähler metric given by

$$Q(\alpha, \beta) = \int_\Sigma \langle \alpha, *\beta \rangle,$$

which clearly shows that the complex structure is compatible with the symplectic form. Since  $\mathcal{M}_T(\Sigma)$  is the symplectic quotient of  $\mathcal{A}(\Sigma)$  with respect to the action of  $T_\partial(\Sigma)$ , cf. Remark 2.11, this induces a compatible complex structure on  $\mathcal{M}_T(\Sigma)$  by Proposition 1.14.

On the other hand,  $\mathcal{M}_T(\Sigma)$  carries a canonical polarization class induced from the one on  $T(\partial\Sigma)$  by Lemma 2.4. As in [PS, Sec. 8.11], it follows that the polarization induced by a complex structure lies exactly in this same polarization class. This completes the proof.  $\square$

**Corollary 2.6.** *The homotopy groups of  $\mathcal{M}_T(\Sigma)$  are given by*

$$\pi_0(\mathcal{M}_T(\Sigma)) \cong H^2(\Sigma, \partial\Sigma; \Lambda), \quad \pi_1(\mathcal{M}_T(\Sigma)) \cong H^1(\Sigma, \partial\Sigma; \Lambda),$$

and  $\pi_n(\mathcal{M}_T(\Sigma)) = 0$  for  $n \geq 2$ .

Finally, we come to the identification of  $\mathcal{M}_T(\Sigma)$  as the moduli space of holomorphic bundles:

**Theorem 2.7** (cf. [Do]). *For  $\Sigma$  complex,  $\mathcal{M}_T(\Sigma)$  is isomorphic to the moduli space of holomorphic  $T_{\mathbb{C}}$ -bundles with a boundary framing, and there is an isomorphism*

$$\mathcal{M}_T(\Sigma) \cong T_{\mathbb{C}}(\partial\Sigma)/T_{\mathbb{C}}^{\Sigma},$$

where  $T_{\mathbb{C}}^{\Sigma} \subseteq T_{\mathbb{C}}(\partial\Sigma)$  is the closed subgroup of loops that admit a holomorphic extension to the surface  $\Sigma$ .

*The prequantum line bundle.* So far we have constructed the moduli space  $\mathcal{M}_T(\Sigma)$  as an affine symplectic manifold. Next, we will consider the set of isomorphism classes of prequantum line bundles over  $\mathcal{M}_T(\Sigma)$ . Denote this set by  $\text{PQ}(\Sigma)$ .

Any prequantum line bundle  $L \in \text{PQ}(\Sigma)$  defines a central extension of the loop group  $T(\partial\Sigma)$ . Indeed, since  $T(\partial\Sigma)$  acts in a Hamiltonian fashion, its action preserves the isomorphism class of  $L$ , and the covering automorphisms define a central extension of  $T(\partial\Sigma)$ . The Lie algebra cocycle of this central extension is given by the symplectic form, and it follows from Stokes' theorem that

$$\begin{aligned} \omega(v_{\xi}, v_{\eta}) &= \int_{\Sigma} \langle d\xi, d\eta \rangle \\ &= \int_{\partial\Sigma} \langle \xi, d\eta \rangle \end{aligned}$$

for  $\xi, \eta \in \Omega^0(\Sigma, \mathfrak{t}) = \text{Lie}(T(\Sigma))$  with generating vector fields  $v_{\xi}, v_{\eta}$ . Since the right hand side is exactly the fundamental cocycle (2.4), the induced central extension must be isomorphic to the central extension of §2.2 associated to the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{t}$ .

**Proposition 2.8.** *There is a short exact sequence*

$$0 \rightarrow H_1(\Sigma, \partial\Sigma; \hat{\Lambda}) \rightarrow \text{PQ}(\Sigma) \rightarrow H_0(\Sigma; \hat{A}) \rightarrow 0.$$

*Proof.* Notice that  $H^0(\Sigma, T)$  acts trivially on  $\mathcal{M}_T(\Sigma)$  via the restriction morphism  $H^0(\Sigma, T) \rightarrow H^0(\partial\Sigma, T)$ , and the level  $\ell$ -central extension of  $T(\partial\Sigma)$  is canonically trivial over  $H^0(\partial\Sigma, T)$ . Therefore, any equivariant line bundle determines a character in  $\lambda \in H_0(\Sigma; \Lambda^{\circ})$ ; the one-dimensional representation by which  $H^0(\Sigma, T)$  acts on the fibers over the identity component. Then  $H^0(\Sigma, T)$  will act over the connected component labeled by  $\chi \in H_0(\Sigma; \Lambda)$  by  $\lambda + \chi \in H_0(\Sigma; \Lambda^{\circ})$ . It follows that two isomorphic prequantum line bundles at the same level determine classes in  $H_0(\Sigma; \Lambda^{\circ})$  that differ only by an element in  $H_0(\Sigma; \Lambda)$ . This defines the second map. It is clearly surjective as one can tensor any prequantum line bundle with a representation of  $H^0(\Sigma, T)$ .

It follows from Lemma 2.4 that the tensor product  $L_1 \otimes L_2^*$  of any two prequantum line bundles at level  $\ell$  carries a canonical  $T(\partial\Sigma)$ -action and descends to a  $H^0(\Sigma, T)$ -equivariant flat line bundle over  $H^1(\Sigma, T)$ . When the  $H^0(\Sigma, T)$  representation is trivial, this is just a flat line bundle over the affine manifold  $H^1(\Sigma, T)$  and its isomorphism class is therefore in

$$\begin{aligned} \text{Hom}(\pi_1(H^1(\Sigma, T)), \mathbb{T}) &\cong \text{Hom}(H^1(\Sigma; \Lambda), \mathbb{T}) \\ &\cong H_1(\Sigma, \partial\Sigma; \hat{\Lambda}). \end{aligned}$$

In other words, the kernel of the map  $\text{PQ}(\Sigma) \rightarrow H_0(\Sigma; A)$  forms a torsor over the compact abelian group  $H_1(\Sigma, \partial\Sigma; \hat{\Lambda})$ . The exact sequence follows by acting on the canonical prequantum line bundle constructed below.  $\square$

**Remark 2.9.** For odd levels, the corresponding statement is that there is an exact sequence

$$0 \rightarrow H_1(\Sigma, \partial\Sigma; \mathbb{Z}_2) \times H_1(\Sigma, \partial\Sigma; \hat{\Lambda}) \rightarrow \text{PQ}(\Sigma) \rightarrow H_0(\Sigma; \hat{A}) \rightarrow 0.$$

The factor  $H_1(\Sigma, \partial\Sigma, \mathbb{Z}_2)$  should be thought of as the space of isomorphism classes of spin structures on  $\Sigma$ . In the construction of the basic prequantum line bundle this choice of spin structure is needed to define a certain lifting cocycle in the definition of the line bundle.

Following [MW], we will construct a prequantum line bundle  $L(\Sigma) \rightarrow \mathcal{M}_T(\Sigma)$  which carries an action of a central extension of  $T(\partial\Sigma)$ , covering the action on  $\mathcal{M}_T(\Sigma)$ . On  $\mathcal{A}(\Sigma)$ , consider the trivial line bundle  $L := \mathcal{A}(\Sigma) \times \mathbb{C}$  with its canonical metric and connection

$$\nabla_\alpha(s)(A) = ds(A) + \frac{i}{2} \int_\Sigma \langle \alpha, A \rangle,$$

for  $\alpha \in T_A \mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{t})$ , and  $s : \mathcal{A}(\Sigma) \rightarrow \mathbb{C}$  a smooth section. One easily finds that  $F_\nabla = -i\omega$ , i.e.,  $(L, \nabla)$  defines a prequantum line bundle on  $\mathcal{A}(\Sigma)$ . The group cocycle  $c : T(\Sigma) \times T(\Sigma) \rightarrow \mathbb{C}$  defined by

$$(2.11) \quad c(\varphi_1, \varphi_2) = \exp \left( -i\pi \int_\Sigma \left\langle \varphi_1^{-1} d\varphi_1, d\varphi_2 \varphi_2^{-1} \right\rangle \right)$$

defines a central extension  $\widetilde{T(\Sigma)}$  of  $T(\Sigma)$  which naturally acts on  $L$  by

$$(\varphi, z) \cdot (A, w) = (\varphi \cdot A, \exp \left( i\pi \int_\Sigma \left\langle \varphi^{-1} d\varphi, A \right\rangle \right) zw).$$

Next, observe that the central extension of  $T(\Sigma)$  can be trivialized over the subgroup  $T_\partial(\Sigma)$ : by Stokes' theorem the cocycle (2.11) is canonically trivial over the identity component of  $T_\partial(\Sigma)$ . The induced central extension of the group of components  $\pi_0(T_\partial(\Sigma)) = H^1(\Sigma, \partial\Sigma; \Lambda)$  is defined by the cocycle  $\exp(i\pi Z)$  where  $Z$  is the bilinear form

$$Z(\psi_1, \psi_2) = \langle \psi \cup \psi_2, \Sigma \rangle,$$

for  $\psi_1, \psi_2 \in H^1(\Sigma, \partial\Sigma; \Lambda)$  and the cup-product also includes the inner product on  $\Lambda$ . Since the lattice is assumed to be even, this cocycle is a coboundary: we have

$$b(\psi_1 + \psi_2) = b(\psi_1)b(\psi_2)c(\psi_1, \psi_2),$$

where  $b : H^1(\Sigma, \partial\Sigma; \Lambda) \rightarrow \mathbb{T}$  is the group cochain defined by

$$b(\psi) := \exp i\pi \int_\Sigma B(\psi, \psi),$$

and  $B$  is the bilinear form on  $\Lambda$  which satisfies (2.3). Although the trivialization involves the choice of  $B$ , this can be made uniformly for all surfaces  $\Sigma$ , and all choices lead to isomorphic theories. When the level is odd, this is however no longer the case and to trivialize this cocycle, i.e., lift the action of  $\pi_0(T_\partial(\Sigma))$ , we need to choose a cobounding spin structure on  $\Sigma$ , i.e., an element in  $H_1(\Sigma, \partial\Sigma; \mathbb{Z}_2)$ , see the remark above.

With this understanding, define the line bundle  $L^{[\chi]}(\Sigma) \rightarrow \mathcal{M}_T^{[\chi]}(\Sigma)$  by

$$L^{[\chi]}(\Sigma) := \left( L|_{\mathcal{A}^\chi(\Sigma)} \otimes \mathbb{C}_\chi \right) / T_\partial(\Sigma),$$

for  $[\chi] \in H^2(\Sigma, \partial\Sigma; \Lambda)$ , where  $\mathbb{C}_\chi$  is the one dimensional irreducible representation of  $H^0(\Sigma, T)$  defined by  $\chi$ . Clearly, this line bundle carries an action of the extension

$$1 \rightarrow \mathbb{T} \rightarrow \widetilde{T(\Sigma)} / T_\partial(\Sigma) \rightarrow T(\partial\Sigma) \rightarrow H^2(\Sigma, \partial\Sigma; \Lambda) \rightarrow 0.$$

In fact, this is a central extension of the kernel of the map  $T(\partial\Sigma) \rightarrow H^2(\Sigma, \partial\Sigma; \Lambda)$  given by (2.7). It remains to extend the action to the level  $\ell$  extension (2.2) of  $T(\partial\Sigma)$ . Of course we define

$$L(\Sigma) := \coprod_{[\chi] \in H_0(\Sigma; \Lambda)} L_\Sigma^{[\chi]}$$

as a line bundle over  $\mathcal{M}_T(\Sigma)$ . Since  $H^0(\Sigma, T)$  is Pontryagin dual to  $H_0(\Sigma; \Lambda)$ , there is a unique way to extend the action of  $T(\partial\Sigma)$  on  $\mathcal{M}_T(\Sigma)$  to that of a central extension on  $L(\Sigma)$ .

**2.4. Quantization.** Now that we have defined a line bundle  $L(\Sigma)$  over  $\mathcal{M}_T(\Sigma)$ , we pick a complex structure on  $\Sigma$  and define  $\mathcal{H}_\Sigma$  to be the quantization as defined in Section 1.2 associated to these data. Explicitly, one has

$$\mathcal{H}_\Sigma = \bigoplus_{\chi \in H_0(\Sigma, \Lambda)} \mathcal{H}_{\mathcal{M}_T^\chi(\Sigma)},$$

where the direct sum is taken in the  $L^2$ -sense. From §1 we know that  $\mathcal{H}_\Sigma$  carries an irreducible representation of a generalized Heisenberg group associated to  $\mathcal{M}_T(\Sigma)$ . Let us analyse this group in some more detail. First notice that the linearization of Lemma 2.4 yields an exact sequence

$$0 \rightarrow H^0(\Sigma, \mathfrak{t}) \rightarrow \Omega^0(\partial\Sigma, \mathfrak{t}) \rightarrow V(\Sigma) \rightarrow H^1(\Sigma, \mathfrak{t}) \rightarrow 0,$$

where the fourth map is induced by taking the de Rham cohomology class of a closed differential form, cf. definition (2.10). For each complex structure, Hodge theory provides a splitting of this exact sequence, representing elements in  $H^1(\Sigma, \mathfrak{t})$  be harmonic forms, so that we have

$$(2.12) \quad V(\Sigma) \cong V(\partial\Sigma) \times H^1(\Sigma, \mathfrak{t}) \times \left( H^0(\partial\Sigma, \mathfrak{t}) / H^0(\Sigma, \mathfrak{t}) \right).$$

In fact, this is an isomorphism of symplectic vector spaces if we introduce the following symplectic structure on  $H^1(\Sigma, \mathfrak{t}) \times (H^0(\partial\Sigma, \mathfrak{t}) / H^0(\Sigma, \mathfrak{t}))$ ;

$$\omega(([\alpha_1], [f_1]), ([\alpha_2], [f_2])) := S\left(\epsilon^{-1}([\alpha_1], \epsilon^{-1}[\alpha_2])\right) + \langle i^*[\alpha_1], [f_2] \rangle - \langle i^*[\alpha_2], [f_1] \rangle.$$

Here  $S$  is the intersection form and the maps  $i^*$  and  $\epsilon$  fit into the long exact cohomology sequence of the pair  $(\Sigma, \partial\Sigma)$ :

$$\begin{aligned} 0 \longrightarrow H^0(\Sigma, \mathfrak{t}) \longrightarrow H^0(\partial\Sigma, \mathfrak{t}) \longrightarrow H^1(\Sigma, \partial\Sigma, \mathfrak{t}) \xrightarrow{\epsilon} \\ \xrightarrow{\epsilon} H^1(\Sigma, \mathfrak{t}) \xrightarrow{i^*} H^1(\partial\Sigma, \mathfrak{t}) \xrightarrow{\delta} H^2(\Sigma, \partial\Sigma, \mathfrak{t}) \longrightarrow 0. \end{aligned}$$

Exactness of this sequence implies that the symplectic form above is nondegenerate. Notice that there is a natural isomorphism

$$H^1(\Sigma, \mathfrak{t}) \times \left( H^0(\partial\Sigma, \mathfrak{t}) / H^0(\Sigma, \mathfrak{t}) \right) \cong H^1(\Sigma, \partial\Sigma, \mathfrak{t}) \times \ker(\delta).$$

From this we see that the commutant in this generalized Heisenberg group is given by

$$\begin{aligned}\pi_1(\mathcal{M}_T(\Sigma))^\perp &= H^1(\Sigma, \partial\Sigma; \Lambda)^\perp \\ &\cong H^1(\Sigma; \Lambda^\circ) \times H^0(\partial\Sigma, \mathfrak{t})/H^0(\Sigma, \mathfrak{t}).\end{aligned}$$

The induced Heisenberg extension of this abelian group has

$$Z\left(H^1(\Sigma; \Lambda^\circ) \times \widetilde{(H^0(\partial\Sigma, \mathfrak{t})/H^0(\Sigma, \mathfrak{t}))}\right) = H^1(\Sigma, \partial\Sigma; \Lambda).$$

From all this we see that the generalized Heisenberg group  $\widetilde{A(\Sigma)}$  associated to  $\mathcal{M}_T(\Sigma)$ , together with the loop group, fits into an exact sequence

(2.13)

$$\begin{aligned}1 \longrightarrow H^0(\partial\Sigma, T) \times \widetilde{H_0(\partial\Sigma, \Lambda)} &\longrightarrow \widetilde{T(\partial\Sigma)} \longrightarrow \widetilde{A(\Sigma)} \longrightarrow \\ &\longrightarrow \left(H^1(\Sigma; \Lambda^\circ) \times \widetilde{H^0(\partial\Sigma, \mathfrak{t})/H^0(\Sigma, \mathfrak{t})}\right) \times \left(H^0(\Sigma, T) \times \widetilde{H_0(\Sigma, \Lambda)}\right) \longrightarrow 1.\end{aligned}$$

We therefore have that

$$Z\left(\widetilde{A(\Sigma)}\right) = H^1(\Sigma, \partial\Sigma; \Lambda) \times H^0(\Sigma; A),$$

which naturally fits with the computation of the Picard group in Proposition 2.8. The crucial point, and the connection to the representation theory of loop groups, is the following:

**Proposition 2.10.** *The Hilbert space  $\mathcal{H}_\Sigma$  carries a positive energy representation of  $T(\partial\Sigma)$ .*

*Proof.* As we have seen, the line bundle  $L(\Sigma) \rightarrow \mathcal{M}_T(\Sigma)$  carries a natural action of the central extension of  $T(\partial\Sigma)$  defined by the inner product  $\langle \cdot, \cdot \rangle$  by automorphisms. This central extension therefore acts projectively on the associated space of holomorphic sections. The affine part of  $T(\partial\Sigma)$ , i.e.,  $V(\partial\Sigma)$ , acts via the morphism in the exact sequence (2.13), and will therefore be implemented by a projective unitary representation on  $\mathcal{H}_\Sigma$ . The extension to  $T(\partial\Sigma)$  is projective unitary as well, because the geometric action of the Heisenberg extension of  $H^0(\partial\Sigma, T) \times H_0(\partial\Sigma, \Lambda)$  preserves the  $L(\Sigma)$ -valued measure determined by the complex structure.

Finally let us prove that this representation has positive energy. As observed in the proof of Proposition 2.5, the standard polarization  $V(\partial\Sigma) = V_+(\partial\Sigma) \oplus V_-(\partial\Sigma)$  of  $V(\partial\Sigma)$  induces a polarization class of  $\mathcal{M}_T(\Sigma)$  by Lemma 2.4, which equals the class defined by any complex structure on  $\Sigma$ . But, as in §2.1, this standard polarization yields a positive energy representation, and therefore, it follows from Theorem A.5 that  $\mathcal{H}_\Sigma$  is of positive energy.  $\square$

**Remark 2.11.** Notice that from its definition (2.8) it follows that  $\mathcal{M}_T(\Sigma)$  itself is the symplectic reduction of the affine symplectic vector space  $\mathcal{A}(\Sigma)$  with respect to the action of  $T_\partial(\Sigma)$ . By Stokes' theorem,  $T_\partial(\Sigma) \hookrightarrow \Omega^1(\Sigma, \mathfrak{t})$  is an isotropic subspace, and the moment map given by the curvature, which is linear in the abelian case, is related to the symplectic form (2.6) by (1.4). By Theorem 2.18, we therefore have

$$\mathcal{H}_{\Sigma, \ell} \cong \bigoplus_{\chi \in H_0(\Sigma; \Lambda)} \text{Hom}_{\widetilde{\Omega^1(\Sigma, \mathfrak{t})}} \left( \text{Ind}_{T_\partial(\Sigma)}^{\widetilde{\Omega^1(\Sigma, \mathfrak{t})}}(\mathbb{C}_\chi); \mathcal{H}_{\mathcal{A}(\Sigma)} \right).$$

In fact, the Hilbert space  $\mathcal{H}_{\mathcal{A}(\Sigma)}$  carries a canonical projective unitary representation of the gauge group  $T(\Sigma)$ .

**Example 2.12.** Well known positive energy representations of  $LT$  are special cases of the quantization procedure described above:

- i) Let  $\Sigma = D$  be a disk. Then we have  $\mathcal{M}_T(D) \cong LT/T$ , and the quantization yields  $\mathcal{H}_D = \mathcal{H}_0$ , the basic representation at the given level. In this case, Proposition 2.8 gives  $\text{PQ}(D) = \hat{A}$ , and if we use the line bundle  $L_\varphi$  labeled by  $\varphi \in \hat{A}$ , we obtain all other irreducible representations of  $LT$  as in §2.2.
- ii) For  $\Sigma = C$  an annulus, we had  $\mathcal{M}_T(\Sigma) \cong T_\omega^*LT \cong LT \times \Omega^1(S^1, \mathfrak{t})$ , cf. Example 2.3 iii). Proposition A.13 therefore gives an isomorphism

$$(2.14) \quad \mathcal{H}_C \cong \bigoplus_{\varphi \in \hat{A}} \mathcal{H}_\varphi \otimes \mathcal{H}_\varphi^*$$

of projective  $LT \times LT^{\text{op}}$ -representations.

**Proposition 2.13.** *For any complex structure on  $\Sigma$ , there is an embedding  $\mathcal{M}_T(\Sigma) \hookrightarrow \mathbb{P}(\mathcal{H}_\Sigma)$  of Kähler manifolds. Let  $\Omega_A$  be a unit vector in the ray determined by the class  $[A] \in \mathcal{M}_T^X(\Sigma)$  of a connection  $A \in \mathcal{A}_X(\Sigma)$ . Then*

$$\oint_{\partial\Sigma} \langle i_{\partial\Sigma}^* A, \xi \rangle = i \langle \Omega_A, \pi(\xi) \Omega_A \rangle$$

for all  $\xi \in \Omega^0(\partial\Sigma, \mathfrak{t})$ .

*Proof.* The first part of the Proposition follows at once from Proposition 1.10. It follows that  $\mathcal{M}_T(\Sigma)$  maps into the smooth vectors  $\mathcal{H}_\Sigma^\infty$  for the representation of  $LT$  on  $\mathcal{H}_\Sigma$ . Since the right hand side of the equality is exactly the moment map for the  $LT$  action on  $\mathbb{P}(\mathcal{H}_\Sigma^\infty)$ , the statement follows from the fact that the moment map on  $\mathcal{M}_T^X(\Sigma)$  is unique.  $\square$

**Remark 2.14.** In particular, the class of the trivial line bundle, the canonical base-point of  $\mathcal{M}_T(\Sigma)$ , yields a ray  $[\Omega_\Sigma] \in \mathbb{P}(\mathcal{H}_\Sigma)$ . This gives rise to the following picture: it follows from Corollary 1.7 and Proposition 2.5, that the quantization  $\mathcal{H}_\Sigma$  is independent of the chosen complex structure. One therefore considers  $\mathcal{H}_\Sigma$  as an abstract positive energy representation of  $T(\partial\Sigma)$  with the universal property that any complex structure on  $\Sigma$  yields a point in  $\mathbb{P}(\mathcal{H}_\Sigma)$  giving rise to a holomorphic embedding  $\mathcal{M}_T(\Sigma) \hookrightarrow \mathbb{P}(\mathcal{H}_\Sigma)$ .

**Theorem 2.15.** *The projective unitary representation of  $T(\partial\Sigma)$  on  $\mathcal{H}_\Sigma$  extends to the semi-direct product  $\mathcal{K}_\Sigma \ltimes T(\partial\Sigma)$ , where  $\mathcal{K}_\Sigma$  is an extension of  $\text{Diff}^+(\partial\Sigma)$  by the mapping class group  $\Gamma(\Sigma, \partial\Sigma)$ ;*

$$1 \rightarrow \Gamma(\Sigma, \partial\Sigma) \rightarrow \mathcal{K}_\Sigma \rightarrow \text{Diff}^+(\partial\Sigma) \rightarrow 1,$$

and the semi-direct product is defined by means of the projection to  $\text{Diff}^+(\partial\Sigma)$ . In particular,  $\mathcal{H}_{\Sigma, \ell}$  carries a projective unitary representation of  $\Gamma(\Sigma, \partial\Sigma)$  commuting with  $LT$ .

*Proof.* The natural action of the group  $\text{Diff}^+(\Sigma)$  of orientation-preserving diffeomorphisms of  $\Sigma$  on  $\mathcal{M}_T(\Sigma)$  factors over  $\text{Diff}_0^+(\Sigma, \partial\Sigma)$ , the identity component of the subgroup of diffeomorphism which leave the boundary point-wise fixed. This one can see as follows: a small computation shows that for a flat connection  $A \in \mathcal{A}_F(\Sigma)$ , the infinitesimal action of a vector field  $X$  on  $\Sigma$  equals

$$-L_X A = -d(\iota_X A).$$

But the right hand side is the generating vector field of the gauge action (2.1) for the Lie algebra element  $\iota_X A \in \Omega^1(\Sigma, \mathfrak{t}) = \text{Lie}(T(\Sigma))$ . In view of the definition (2.8) of  $\mathcal{M}_T(\Sigma)$ , this induces an action of

$$\mathcal{K}_\Sigma := \text{Diff}^+(\Sigma) / \text{Diff}_0^+(\Sigma, \partial\Sigma).$$

Clearly, the action of  $\text{Diff}^+(\Sigma)$  preserves the symplectic form (2.6), and one finds a natural map  $\mathcal{K}_\Sigma \rightarrow \text{Aut}_{\text{res}}(\mathcal{M}_T(\Sigma))$ . The desired representation now follows from Prop. 1.13. The exact sequence in statement of the proposition follows immediately from the definition  $\Gamma(\Sigma, \partial\Sigma) := \text{Diff}^+(\Sigma, \partial\Sigma) / \text{Diff}_0^+(\Sigma, \partial\Sigma)$  and the fact that  $\text{Diff}^+(\Sigma) / \text{Diff}^+(\Sigma, \partial\Sigma) \cong \text{Diff}^+(\partial\Sigma)$ .  $\square$

**2.5. Gluing and reduction.** Let  $\Sigma_1$  and  $\Sigma_2$  be two Riemann surfaces with parametrized boundaries. We can use the parametrization to glue  $\Sigma_1$  and  $\Sigma_2$  over a compact 1-manifold  $S$ , of course diffeomorphic to a disjoint union of copies of  $S^1$ . We denote the resulting smooth surface by  $\Sigma = \Sigma_1 \cup_S \Sigma_2$ . Its boundary, which may be empty, inherits a canonical parametrization from  $\Sigma_1$  and  $\Sigma_2$ .

**Proposition 2.16.** (cf. [MW, Thm. 3.5]) *In the situation above, the moduli space  $\mathcal{M}_T(\Sigma)$  is obtained from the moduli spaces associated to  $\Sigma_1$  and  $\Sigma_2$  by symplectic reduction;*

$$\mathcal{M}_T(\Sigma) \cong (\mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)) // T(S).$$

*The same holds true for the prequantum line bundle;*

$$L(\Sigma) \cong (L(\Sigma_1) \times L(\Sigma_2)) // T(S).$$

*Proof.* First consider the reduction of two connected components  $\mathcal{M}_T^{\chi_1}(\Sigma_1)$  and  $\mathcal{M}_T^{\chi_2}(\Sigma_2)$  with respect to the subgroup  $T(S)_0$  consisting of the kernel of the natural map  $T(S) \rightarrow H_0(\Sigma_1; \Lambda) \oplus H_0(\Sigma_2; \Lambda)$  induced by the exact sequence (2.7) and the diagonal inclusion  $T(S) \hookrightarrow T(\partial\Sigma_1) \times T(\partial\Sigma_2)$ . We can choose de Rham representative  $\chi_1 \in \Omega_\Lambda^2(\Sigma_1, \mathfrak{t})$  and  $\chi_2 \in \Omega_\Lambda^2(\Sigma_2, \mathfrak{t})$  which are the pull-back of  $\chi \in \Omega_\Lambda^2(\Sigma, \mathfrak{t})$ . Since the moment map of the action of  $T(S)$  on  $\mathcal{M}_T(\Sigma_1)$  and  $\mathcal{M}_T(\Sigma_2)$  is given by restriction of connections to the boundary, cf. (2.9), the zero locus of the moment map on  $\mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)$  corresponding to the diagonal  $T(S)_0$ -action is given by the set  $([A_1], [A_2]) \in \mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)$  satisfying

$$A_1|_{\partial\Sigma_1} = A_2|_{\partial\Sigma_2}.$$

As in [MW, Thm 3.5], one shows that this defines, up to the  $T(S)_0$ -action, a unique gauge equivalence class of a smooth connection  $[A]$  on  $\Sigma$  with  $F_A = \chi$ . This defines a smooth symplectomorphism

$$(\mathcal{M}_T^{\chi_1}(\Sigma_1) \times \mathcal{M}_T^{\chi_2}(\Sigma_2)) // T(S)_0 \xrightarrow{\cong} \mathcal{M}_T^\chi(\Sigma).$$

Recall that by Corollary 2.6, the  $\chi$  represent homology classes in  $H_0(\Sigma; \Lambda)$  which label the connected components of  $\mathcal{M}_T(\Sigma)$ , and the above map induces on the level of  $\pi_0$  the natural morphism

$$i_* : H_0(\Sigma_1; \Lambda) \oplus H_0(\Sigma_2; \Lambda) \rightarrow H_0(\Sigma; \Lambda)$$

which fits into the Mayer–Vietoris sequence, cf. Remark 2.17 below. By exactness of this sequence, its kernel consist therefore of the image of  $H_0(S; \Lambda)$  in  $H_0(\Sigma_1; \Lambda) \oplus H_0(\Sigma_2; \Lambda)$ , and since  $\pi_0(T(S)) = H_0(S; \Lambda)$ , one finds that  $\ker i_* = T(S) / T(S)_0$ .



Since the group of components of the boundary gauge group acts by identifying different components of the moduli spaces, we see that the induced map

$$(\mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)) // T(S) \rightarrow \mathcal{M}_T(\Sigma),$$

which is a surjective symplectic submersion as shown above, is in fact injective. This proves the statement for the moduli spaces. The gluing of the prequantum line bundle is proved in similar fashion; we omit the details.  $\square$

**Remark 2.17.** As affine symplectic manifolds, the modelling spaces of the manifolds in this proposition are related by symplectic reduction of vector spaces:

$$(2.15) \quad V(\Sigma) \cong (V(\Sigma_1) \times V(\Sigma_2)) // V(S).$$

Using Corollary 2.6 and applying Poincaré–Lefschetz duality, the exact sequence (1.5) of fundamental groups amounts to the Mayer–Vietoris sequence in homology:

$$\begin{aligned} 0 \rightarrow H_1(S; \Lambda) \rightarrow H_1(\Sigma_1; \Lambda) \oplus H_1(\Sigma_2; \Lambda) \rightarrow H_1(\Sigma; \Lambda) \rightarrow \\ \rightarrow H_0(S; \Lambda) \rightarrow H_0(\Sigma_1; \Lambda) \oplus H_0(\Sigma_2; \Lambda) \rightarrow H_0(\Sigma; \Lambda) \rightarrow 0. \end{aligned}$$

(In comparison to (1.5), there is an extra term corresponding to  $\pi_1(T(S)) = H^0(S, \Lambda)$  since the loop group  $T(S)$  is not quite affine; it contains a factor  $H^0(S, T)$ .)

The quantum version of this theorem is as follows:

**Theorem 2.18.** *Let  $\Sigma = \Sigma_1 \cup_S \Sigma_2$  be obtained by gluing  $\Sigma_1$  and  $\Sigma_2$  along a common boundary. There is a canonical isomorphism*

$$\mathcal{H}_\Sigma \cong \text{Hom}_{\widetilde{T(S) \times T(S)}_{op}} \left( \text{Ind}_{T(S)}^{\widetilde{T(S) \times T(S)}_{op}}(\mathbb{C}), \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \right).$$

*Proof.* As usual, we decompose  $T(S) = V(S) \times H^0(S; \Lambda) \times H^0(S, T)$ . Let us first consider the affine part of  $T(S)$ , namely  $V(S)$ . It acts on  $\mathcal{H}_\Sigma$  by the morphism in the exact sequence (2.13). Introduce the affine symplectic manifold

$$\mathcal{M}_\Sigma^p := (\mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)) // V(S),$$

and denote by  $\mathcal{H}_\Sigma^p$  its quantization. By Theorem 1.19 we have

$$\mathcal{H}_\Sigma^p \cong \text{Hom}_{A(\Sigma_1 \sqcup \Sigma_2)} \left( \text{Ind}_{V(S) \times Z(A(\Sigma_1 \sqcup \Sigma_2))}^{A(\Sigma_1 \sqcup \Sigma_2)} (L(\Sigma)_e), \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \right),$$

where, as before,  $Z(A(\Sigma)) = H^1(\Sigma, \partial\Sigma; \Lambda) \times H^0(\Sigma, A_\ell)$ . Notice that the prequantum line bundle  $L(\Sigma)$  is such that the induced central extension of  $Z(A(\Sigma))$  is canonically trivial, i.e.,  $L(\Sigma)_e$  is just the trivial representation. The splitting (2.12) induces a splitting  $A(\Sigma_1 \sqcup \Sigma_2) \cong V(S) \times H(\Sigma, S)$  compatible with the cocycles defining the Heisenberg group, with  $H(\Sigma, S)$  containing the center of  $A(\Sigma_1 \sqcup \Sigma_2)$ . With this splitting we can decompose the induced representation as

$$\begin{aligned} \text{Ind}_{V(S) \times Z(A(\Sigma_1 \sqcup \Sigma_2))}^{A(\Sigma_1 \sqcup \Sigma_2)}(\mathbb{C}) &= \text{Ind}_{Z(A(\Sigma_1 \sqcup \Sigma_2))}^{\widetilde{H(\Sigma, S)}}(\mathbb{C}) \otimes \text{Ind}_{V(S)}^{\widetilde{V(S) \times \widetilde{V(S)}}_{op}}(\mathbb{C}) \\ &\cong \mathcal{H}_{H(\Sigma, S), 0} \otimes \mathcal{H}_{H(\Sigma, S), 0}^* \otimes \text{Ind}_{V(S^1)}^{\widetilde{V(S_1) \times \widetilde{V(S)}}_{op}}(\mathbb{C}), \end{aligned}$$

by Theorem A.11. Here  $\mathcal{H}_{H(\Sigma, S), 0}$  is the unique irreducible representation of the Heisenberg group of  $H(\Sigma, S)$  where the center acts trivially. Taking the intertwiner spaces, we therefore obtain

$$\mathcal{H}_{\Sigma}^p \cong \text{Hom}_{\widetilde{V(S)} \times \widetilde{V(S)}_{op}} \left( \text{Ind}_{V(S)}^{\widetilde{V(S)} \times \widetilde{V(S)}_{op}} (\mathbb{C}), \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \right).$$

This takes care of the affine part of the loop group. Next we consider the action of the central extension of  $H^0(S; \Lambda) \times H^0(S, T)$ . Before proceeding with the proof, consider the following:

**Lemma 2.19.** *Let  $\mathcal{H}$  carry a representation of  $(\widetilde{\Lambda \times T}) \times (\widetilde{\Lambda \times T})_{op}$ . There is a canonical isomorphism*

$$\text{Hom}_{(\widetilde{\Lambda \times T}) \times (\widetilde{\Lambda \times T})_{op}} \left( L^2(\Lambda \times T), \mathcal{H} \right) \xrightarrow{\cong} \mathcal{H}^{T \times T}.$$

*Proof.* Let  $f_0 \in L^2(\Lambda \times T)$  be the function which is 1 on  $\{0\} \times T$  and zero else. This defines a map

$$\text{Hom}_{(\widetilde{\Lambda \times T}) \times (\widetilde{\Lambda \times T})_{op}} \left( L^2(\Lambda \times T), \mathcal{H}_1 \otimes \mathcal{H}_2 \right) \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$$

by  $\psi \mapsto \psi(f_0)$ . This map is clearly an isometry because

$$\begin{aligned} \|\psi(f_0)\|^2 &= \langle \psi(f_0), \psi(f_0) \rangle \\ &= \langle \psi^* \psi(f_0), f_0 \rangle \\ &= \psi^* \psi \langle f_0, f_0 \rangle \\ &= \|\psi\|^2, \end{aligned}$$

where we have used Schur's lemma. Since  $f_0$  is invariant under the diagonal  $T \times T$ -action, it maps into  $\mathcal{H}^{T \times T}$ . It is easy to see that it is surjective onto this space, it therefore induces the unitary isomorphism of the lemma.  $\square$

By Proposition 2.16 and reduction in stages there are isomorphisms

$$\begin{aligned} \mathcal{M}_T(\Sigma) &\cong (\mathcal{M}_T(\Sigma) \times \mathcal{M}_T(\Sigma_2)) // T(S) \\ &\cong \left( ((\mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)) // V(S)) // H^0(S, T) \right) / H^1(S; \Lambda) \\ &= \left( (\mathcal{M}_T^p(\Sigma)) // H^0(S, T) \right) / H^1(S; \Lambda). \end{aligned}$$

Motivated by the Lemma above, we will show that

$$(2.16) \quad \left( \mathcal{H}_{\Sigma}^p \right)^{H^0(S \sqcup S, T)} \cong \mathcal{H}_{\Sigma}.$$

There are various ways of doing this, but we choose the following. The Hilbert space  $\mathcal{H}_{\Sigma}^p$  carries an irreducible representation of a Heisenberg group  $\widetilde{A(\Sigma, S)}$  which can be split as

$$\begin{aligned} \widetilde{A(\Sigma, S)} &\cong \widetilde{V(\partial \Sigma)} \times (H^1(\Sigma_1 \sqcup \Sigma_2, \mathfrak{t}) \times (H^0(\partial(\Sigma_1 \sqcup \Sigma_2), \mathfrak{t}) / H^0(\Sigma_1 \sqcup \Sigma_2, \mathfrak{t}))) \\ &\quad \times H^0(\Sigma_1 \sqcup \Sigma_2, T) \times H_0(\Sigma_1 \sqcup \Sigma_2; \Lambda) \end{aligned}$$

This follows by the same reasoning that lead to the exact sequence (2.13), and the cocycle on the right hand side is as explained there. Now, since  $H^0(S \sqcup S, T)$  is

compact, to determine the subspace of invariants of  $\mathcal{H}_\Sigma^p$ , it suffices to consider the dense subspace annihilated by its Lie algebra  $H^0(S \sqcup S, \mathfrak{t})$ . But this Lie algebra acts on  $\mathcal{H}_\Sigma^p$  via a map to  $A(\Sigma, S)$  whose image is an isotropic subgroup, so we see that the  $H^0(S \sqcup S, \mathfrak{t})$ -invariants of  $\mathcal{H}_\Sigma^p$  carry an irreducible representation of the induced Heisenberg extension of  $H^0(S \sqcup S, \mathfrak{t})^\perp / H^0(S, \sqcup S, \mathfrak{t})$ . We now claim that

$$H^0(S \sqcup S, \mathfrak{t})^\perp / H^0(S, \sqcup S, \mathfrak{t}) \cong \widetilde{A(\Sigma)},$$

proving the isomorphism (2.16). To see this, notice that  $H^0(S \sqcup S, \mathfrak{t})$  embeds into  $A(\Sigma)$  in two ways: first via the natural morphism  $H^0(S \sqcup S, \mathfrak{t}) \rightarrow H^0(\partial(\Sigma_1 \sqcup \Sigma_2, \mathfrak{t}) / H^0(\Sigma_1 \sqcup \Sigma_2, \mathfrak{t}))$ , whose kernel equals the image of the restriction map  $H^0(\Sigma_1 \sqcup \Sigma_2, \mathfrak{t}) \rightarrow H^0(S \sqcup S, \mathfrak{t})$ . Second, exactly this image maps via the exponential mapping to  $H^0(\Sigma_1 \sqcup \Sigma_2, T)$ . With this the result follows.  $\square$

**Remark 2.20.** For the diagonal embedding  $LT \hookrightarrow \widetilde{LT} \times \widetilde{LT}_{op}$ , it follows from Theorem A.11 that

$$(2.17) \quad \text{Ind}_{LT}^{\widetilde{LT} \times \widetilde{LT}_{op}}(\mathbb{C}) = \bigoplus_{\lambda \in \hat{A}} \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda^*.$$

This also explains the precise nature of the inner-product used on the right hand side of Theorem 2.18: it is the sum of the usual inner-product on intertwiners from the irreducible  $\widetilde{LT} \times \widetilde{LT}_{op}$ -representations  $\mathcal{H}_\lambda \otimes \mathcal{H}_\lambda^*$ .

### 3. CONFORMAL FIELD THEORY

**3.1. A finite Heisenberg group and its representations.** In this section we will construct a certain finite Heisenberg group whose representation theory controls the structure of the conformal field theories of this paper. Its construction is completely topological: Let  $\Sigma$  be an oriented surface with compact oriented boundaries, and consider the homology group  $H_1(\Sigma; A)$ , which is finite. Let  $\ell \in \mathbb{N}$  be the largest integer such that the inner product  $\langle \cdot, \cdot \rangle$ , restricted to  $\Lambda$  takes values in  $\ell\mathbb{Z}$ . The induced bilinear form

$$\langle \cdot, \cdot \rangle : A \times A \rightarrow \mathbb{Z} / \ell\mathbb{Z},$$

together with the intersection form determine a  $\mathbb{Z} / \ell\mathbb{Z}$ -valued antisymmetric form, denoted by  $S$ . Associated to  $S$  is a central extension

$$(3.1) \quad 0 \rightarrow \mathbb{Z} / \ell\mathbb{Z} \rightarrow \widetilde{H_1(\Sigma; A)} \rightarrow H_1(\Sigma; A) \rightarrow 0,$$

defined as  $\widetilde{H_1(\Sigma; A)} = H_1(\Sigma; A) \times \mathbb{Z}_\ell$  with product given by

$$(X, m) \cdot (Y, n) = (X + Y, m + n + S(X \cap Y)).$$

This group is called the finite Heisenberg group associated to the pair  $(H_1(\Sigma; A), S)$ . The reason for this terminology becomes more clear when thinking of  $\mathbb{Z}_\ell$  as the group of  $\ell$ 'th roots of unity. Notice that both  $H_1(\Sigma; A)$  and its Heisenberg extension are finite groups. The following proposition classifies and constructs the irreducible representations of this group in which the central  $\mathbb{Z} / \ell\mathbb{Z}$  acts by roots of unity.

**Proposition 3.1.** *Irreducible representations of  $\widetilde{H_1(\Sigma; A)}$  in which the central  $\mathbb{Z}_\ell$  acts via the standard representation are classified and constructed as follows:*

i) Any representation  $\mathcal{H}$  naturally decomposes as

$$\mathcal{H} = \bigoplus_{\substack{\vec{\lambda} \in H^1(\partial\Sigma; \hat{A}) \\ \delta(\vec{\lambda})=0}} \mathcal{H}_{\hat{\Sigma}} \otimes \mathbb{C}_{\vec{\lambda}},$$

where  $\mathcal{H}_{\hat{\Sigma}}$  is a unitary representation of  $H_1(\hat{\Sigma}; A)$ , the Heisenberg group associated to the surface  $\hat{\Sigma}$ , obtained by gluing disks to the boundaries of  $\Sigma$ , and  $\delta$  is the connecting homomorphism in the long exact cohomology sequence associated to the pair  $(\Sigma, \partial\Sigma)$ ,

ii) the irreducible representations are classified by cohomology classes  $\vec{\lambda} \in H^1(\partial\Sigma; \hat{A})$  such that  $\delta(\vec{\lambda}) = 0$ . They are finite dimensional.

*Proof.* Consider the long exact homology sequence associated to the couple  $(\Sigma, \partial\Sigma)$ :

$$\dots \longrightarrow H_1(\partial\Sigma; A) \xrightarrow{i_*} H_1(\Sigma; A) \longrightarrow H_1(\Sigma, \partial\Sigma; A) \longrightarrow \dots$$

The image of  $H_1(\partial\Sigma; A)$  is exactly the kernel of the intersection product  $S$  and therefore the center of the Heisenberg extension (3.1) equals  $\text{Im}(i_*) \times \mathbb{Z}_\ell$ . The Pontryagin dual of the first factor coming from  $H_1(\partial\Sigma; A)$  is naturally isomorphic to the group of elements  $\vec{\lambda} \in H^1(\partial\Sigma; \hat{A})$  such that  $\delta(\vec{\lambda}) = 0$ , where the last condition accounts for the effect that the group acts via the homomorphism  $H_1(\partial\Sigma; A) \rightarrow H_1(\Sigma; A)$ . In view of Theorem A.5, this gives ii). Furthermore, in any isotypical summand of a representation  $\mathcal{H}$ , we can mod out the center to obtain a unitary representation of the nondegenerate Heisenberg group

$$H^1(\Sigma; A) / \text{Im}(i_*) \cong H^1(\hat{\Sigma}; A).$$

Decomposing the representation  $\mathcal{H}$  under the action of the center, the result now follows.  $\square$

**3.2. The determinant line bundle.** Let  $\mathcal{E}$  be the “category” of Riemann surfaces as in [S3]: its objects are the natural numbers  $\mathbb{N}$  and morphisms consist of equivalence classes of complex Riemann surfaces with boundaries parametrized by a disjoint union of copies of  $S^1$ . The space of morphisms  $\mathcal{E}(m, n)$  decomposes as a disjoint union

$$\mathcal{E}(m, n) = \coprod_{g \geq 0} \mathcal{E}(g, m, n)$$

according to the “genus” of the surfaces, where each  $\mathcal{E}(g, m, n)$  is a complex manifold. Composition in the category is induced by gluing Riemann surfaces using the boundary parametrization and this defines holomorphic maps

$$\mathcal{E}(m, n) \times \mathcal{E}(n, k) \rightarrow \mathcal{E}(m, k).$$

Finally notice that  $\mathcal{E}$  is not a category in the usual sense, since it has no unit morphisms.

Associated to each connected component  $\mathcal{E}(g, m, n)$  is the unique affine symplectic manifold  $\mathcal{M}_T(\Sigma)$ , where  $\Sigma$  is a model surface of genus  $g$  with  $m + n$  boundary components. (In this case we also write  $\mathcal{E}(\Sigma) := \mathcal{E}(g, m, n)$ .) Indeed its symplecto-geometric structure does not depend on the choice of a complex structure on  $\Sigma$ . By Proposition 2.5, a complex structure induces a polarization of  $\mathcal{M}_T(\Sigma)$  and this leads to a map

$$\mathcal{E}(\Sigma) \rightarrow \mathcal{J}(V(\Sigma))$$

which is in fact a holomorphic embedding. Using Theorem 1.11, one finds that the tensor product

$$\mathrm{Det}_{V(\Sigma)}^{-1/2} \otimes \mathcal{H}_\Sigma$$

forms a flat holomorphic Hilbert bundle over  $\mathcal{E}(g, m, n)$ . Recall that the line bundle  $\mathrm{Det}_{V(\Sigma)}^{-1/2}$  is the dual of the line bundle formed by the ray of the unit element under the canonical embedding  $\mathcal{M}_T(\Sigma) \hookrightarrow \mathbb{P}\mathcal{H}_\Sigma$  of Proposition 2.13. For the following, recall the notion of a central extension of  $\mathcal{E}$  and the definition of the determinant line bundle  $\mathrm{Det}_\Sigma$  [S3].

**Theorem 3.2.** *The line bundle*

$$\mathrm{Det}_{V(\Sigma)}^{1/2} \rightarrow \mathcal{E}(g, m, n)$$

*forms a holomorphic central extension of the category  $\mathcal{E}$ , and is canonically isomorphic to the determinant line bundle  $\mathrm{Det}_\Sigma^c$  with  $c = \ell \dim(\mathfrak{t})$ . Therefore, the tensor product*

$$\mathrm{Det}_\Sigma^{-c} \otimes \mathcal{H}_{\Sigma, \ell}$$

*forms a flat Hilbert bundle over  $\mathcal{E}(\Sigma)$ , compatible with gluing.*

*Proof.* Clearly,  $\mathrm{Det}_{V(\Sigma)}^{-1/2}$  forms a holomorphic line bundle over  $\mathcal{E}(g, m, n)$ , since it is the pull-back of a holomorphic line bundle over  $\mathcal{J}(V(\Sigma))$ . To define a central extension, there should be isomorphisms

$$(3.2) \quad \mathrm{Det}_{V(\Sigma_1)}^{-1/2} \boxtimes \mathrm{Det}_{V(\Sigma_2)}^{-1/2} \rightarrow \mathrm{Det}_{V(\Sigma_1 \cup_S \Sigma_2)}^{-1/2},$$

when gluing  $\Sigma_1$  and  $\Sigma_2$  along  $S$ . By the isomorphism (2.15), the vector spaces  $V(\Sigma_1)$ ,  $V(\Sigma_2)$  and  $V(\Sigma)$  are related by symplectic reduction along  $V(S)$ . The linearization of Theorem 2.18 yields the isomorphism

$$(3.3) \quad \mathcal{H}_{V(\Sigma)} \cong \mathrm{Hom}_{\widetilde{V(S)} \times \widetilde{V(S)}_{op}} \left( \mathrm{Ind}_{V(S)}^{\widetilde{V(S)} \times \widetilde{V(S)}_{op}}(\mathbb{C}), \mathcal{H}_{V(\Sigma_1 \sqcup \Sigma_2)} \right).$$

Proposition A.13 gives the canonical isomorphism

$$\mathrm{Ind}_{V(S)}^{\widetilde{V(S)} \times \widetilde{V(S)}_{op}}(\mathbb{C}) = L^2(V(S)^*, d\mu) \cong \mathcal{H}_{V(S)} \otimes \mathcal{H}_{V(S)}^*,$$

which is canonically independent of the polarizations, i.e., defines a flat Hilbert bundle over  $\mathcal{J}(V(\Sigma))$ . But this implies that the isomorphism 3.3 is canonical, i.e., defines an isomorphism of projectively flat Hilbert bundles over  $\mathcal{J}(V(\Sigma))$ . This defines the isomorphism (3.2) of line bundles as the tensor product with the left and right hand side yields a flat Hilbert bundle by Proposition 1.11.

Concluding,  $\mathrm{Det}_{V(\Sigma)}^{1/2}$  defines a central extension of the complex cobordism category  $\mathcal{E}$ , or in other words a one-dimensional holomorphic modular functor (see below). But these central extensions can be classified [S3, K]: any such central extension is isomorphic to a tensor power of the determinant line bundle of the  $\bar{\partial}$ -operator on  $\Sigma$ . This tensor power is determined by the induced central extension of the diffeomorphism group. For the loop group of a torus, this is known to be  $\ell \dim(\mathfrak{t})$  [PS].  $\square$

**Remark 3.3.** For odd levels, one actually finds a central extension of  $\mathcal{E}_{spin}$ , the complex cobordism category of one-manifolds and surfaces equipped with a spin structure. The forgetful functor turns this category into a finite covering of  $\mathcal{E}$ . The spin structure on the morphisms comes in when constructing the prequantum line bundle, see Remark 2.9.

**3.3. A unitary modular functor.** Let  $\Sigma$  be a Riemann surface with parametrized boundaries. A labeling of  $\Sigma$  is given by an element  $\vec{\lambda} \in H^1(\partial\Sigma; \hat{A})$ . The division of the boundary components of  $\Sigma$  into “incoming” and “outgoing” splits the labeling set as  $\vec{\lambda} = (\vec{\lambda}_{in}, \vec{\lambda}_{out})$ . Recall from §2.2 that the elements in  $H^1(\partial\Sigma, \hat{A})$  also label the irreducible positive energy representations of  $T(\partial\Sigma)$ . We therefore define

$$(3.4) \quad E(\Sigma, \vec{\lambda}) := \text{Hom}_{\widetilde{T(\partial\Sigma)}} \left( \mathcal{H}_{\vec{\lambda}_{in}}^* \otimes \mathcal{H}_{\vec{\lambda}_{out}}, \mathcal{H}_{\Sigma} \right).$$

Of course, these are just the multiplicity spaces of the representation of  $\widetilde{T(\partial\Sigma)}$  on  $\mathcal{H}_{\Sigma}$  and with this definition there is a canonical decomposition

$$\mathcal{H}_{\Sigma} = \bigoplus_{\vec{\lambda} \in H^1(\partial\Sigma; \hat{A})} E(\Sigma, \vec{\lambda}) \otimes \mathcal{H}_{\vec{\lambda}_{in}}^* \otimes \mathcal{H}_{\vec{\lambda}_{out}}.$$

Let us first observe the following:

**Proposition 3.4.**  $E(\Sigma, \vec{\lambda})$  carries an irreducible representation of  $H_1(\widetilde{\Sigma}; A)$  with center  $H_1(\partial\Sigma, A)$  acting via the character  $\vec{\lambda}$ . In particular,  $E(\Sigma, \vec{\lambda})$  is finite dimensional.

*Proof.* By the decomposition (2.17), we have a canonical isomorphism

$$E(\Sigma, \vec{\lambda}) \cong \text{Hom}_{\widetilde{T(\partial\Sigma) \times T(\partial\Sigma)_{op}}} \left( \text{Ind}_{\widetilde{T(\partial\Sigma)}}^{\widetilde{T(\partial\Sigma) \times T(\partial\Sigma)_{op}}} (\mathbb{C}), \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\vec{\lambda}_{in}} \otimes \mathcal{H}_{\vec{\lambda}_{out}}^* \right).$$

Let  $D$  be the disjoint union of disks needed to close  $\Sigma$  to  $\hat{\Sigma}$  by gluing over  $\partial\Sigma$ . Recall that  $\mathcal{H}_{\vec{\lambda}}$  is the quantization of the affine symplectic manifold  $\mathcal{M}_T(D)$  using the line bundle  $L_{\vec{\lambda}}$ . Therefore, by exactly the same reasoning as in Theorem 2.18, we see that  $E(\Sigma, \vec{\lambda})$  is the quantization of

$$\mathcal{M}_T(\hat{\Sigma}) \cong (\mathcal{M}_T(\Sigma) \times \mathcal{M}_T(D)) // T(\partial\Sigma),$$

with prequantum line bundle

$$L(\Sigma, \vec{\lambda}) := \left( (L(\Sigma) \otimes L_{\vec{\lambda}}) |_{\mu^{-1}(\mathcal{O}(\partial\Sigma))} \right) / T(\partial\Sigma).$$

The moduli space  $\mathcal{M}_T(\hat{\Sigma})$  is affine symplectic and modelled on

$$T_{[A]} \mathcal{M}_T(\hat{\Sigma}) = H^1(\hat{\Sigma}; \mathfrak{t}).$$

As observed in Example 2.3 *i*), the unit component of  $\mathcal{M}_T(\hat{\Sigma})$  is compact and therefore its quantization will be finite dimensional. By Corollary 2.6,

$$\pi_1(\mathcal{M}_T(\hat{\Sigma})) = H^1(\hat{\Sigma}, \Lambda),$$

and thus the quantization will carry an irreducible representation of the Heisenberg extension of  $H^1(\hat{\Sigma}; \Lambda^\circ)$ , in which the center  $H^1(\hat{\Sigma}; \Lambda)$  acts trivially. Additionally, as is clear from the construction of the line bundle  $L(\Sigma, \vec{\lambda})$ , there is a residual action of  $H^0(\partial\Sigma; A)$  labeled by the character  $\vec{\lambda} \in H^1(\partial\Sigma; \hat{A})$ . By Proposition

3.1, this amounts to an irreducible representation of the Heisenberg extension of  $H_1(\Sigma; A)$ . This completes the proof.  $\square$

**Remark 3.5.** The classification theorem 3.1 involves the condition  $\delta(\vec{\lambda}) = 0$ . This can be seen as follows: the argument above proves that

$$E(\Sigma, \vec{\lambda}) \cong \left( \mathcal{H}_{\mathcal{M}_T(\hat{\Sigma}), L(\Sigma, \vec{\lambda})} \right)^{H^0(\hat{\Sigma}, T)}.$$

This equals of course the quantization of the component of  $\mathcal{M}_T(\hat{\Sigma})$  over which  $H^0(\hat{\Sigma}, T)$  acts on the restriction of the prequantum line bundle  $L(\Sigma, \vec{\lambda})$  by the trivial character. Recall that  $\pi_0(\mathcal{M}_T(\hat{\Sigma})) = H_0(\hat{\Sigma}; \Lambda)$ . Let  $D$  be the union of disks used to obtain  $\hat{\Sigma}$  from  $\Sigma$ . The Mayer–Vietoris sequence gives

$$\dots \longrightarrow H_0(\partial\Sigma; \Lambda^\circ) \longrightarrow H_0(D; \Lambda^\circ) \oplus H_0(\Sigma; \Lambda^\circ) \longrightarrow H_0(\hat{\Sigma}; \Lambda^\circ) \longrightarrow \dots,$$

which determines the character in  $H_0(\hat{\Sigma}; \Lambda^\circ)$  by which  $H^0(\Sigma, T)$  acts: over the connected component determined by  $\nu \in H_0(\Sigma; \Lambda)$  and  $\mu \in H_0(D; \Lambda)$ , this is given by the image of  $(\nu, \vec{\lambda} + \vec{\mu}) \in H_0(D; \Lambda^\circ) \oplus H_0(\Sigma; \Lambda^\circ)$  under the map above. Taking  $H^0(\hat{\Sigma}, T)$ -invariants therefore amounts to considering the connected component  $\mathcal{M}_T^0(\hat{\Sigma})$  with the line bundle  $L(\Sigma, \vec{\lambda})$  subject to the condition  $\delta(\vec{\lambda}) = 0$ .

**Theorem 3.6.** *The assignment  $(\Sigma, \vec{\lambda}) \mapsto E(\Sigma, \vec{\lambda})$  is part of a unitary modular functor, i.e., the vector spaces  $E(\Sigma, \vec{\lambda})$  form holomorphic hermitian vector bundles over the moduli space  $\mathcal{E}(\Sigma)$  such that:*

- i) (Normalization)  $\dim E(\mathbb{CP}^1) = 1$ ,
- ii) (Tensor property)  $E(\Sigma_1 \sqcup \Sigma_2) = E(\Sigma_1) \otimes E(\Sigma_2)$ ,
- iii) (Duality) there is a functorial isomorphism  $E(\overline{\Sigma}, \vec{\lambda}) \cong E(\Sigma, \vec{\lambda})^*$ ,
- iv) (Factorization) when  $\Sigma = \Sigma_1 \cup_S \Sigma_2$ , we have a canonical isomorphism

$$\bigoplus_{\vec{\lambda}_S \in H^1(S; \hat{A})} E(\Sigma_1, \vec{\lambda}_1, \vec{\lambda}_S) \otimes E(\Sigma_2, \vec{\lambda}_S, \vec{\lambda}_2) \xrightarrow{\cong} E(\Sigma, \vec{\lambda}),$$

where the labeling  $\vec{\lambda} = (\vec{\lambda}_1, \vec{\lambda}_2) \in H^1(\partial\Sigma; \hat{A})$  is divided into a partial labeling of  $\Sigma_1$  and  $\Sigma_2$ , which is completed by the  $\vec{\lambda}_S \in H^1(S; \hat{A})$ ,

- v) (Coherency) Let  $c = \ell \dim(\mathfrak{t})$ . The tensor product

$$\tilde{E}(\Sigma, \vec{\lambda}) := E(\Sigma, \vec{\lambda}) \otimes \text{Det}_\Sigma^c$$

is a flat bundle of Hilbert spaces, i.e., carries a flat connection which preserves a positive definite nondegenerate hermitian form, and the connection and inner product are compatible with the isomorphisms of ii), iii) and iv) above.

*Proof.* By the previous proposition, the spaces  $E(\Sigma, \vec{\lambda})$  are finite dimensional. All five properties follow from general results about the quantizations of the moduli spaces  $\mathcal{M}_T(\Sigma)$  proved in the course of the paper:

- i) Present  $\mathbb{CP}^1$  as the result of gluing two disks  $D$  to each other. Since quantization of  $\mathcal{M}_T(D)$  yields the basic representation at level  $\ell$ , the property follows at once from the definition (3.4) by Schur’s lemma.
- ii) This follows from the canonical isomorphism  $\mathcal{M}_T(\Sigma_1 \sqcup \Sigma_2) \cong \mathcal{M}_T(\Sigma_1) \times \mathcal{M}_T(\Sigma_2)$ , and the “tensor property” of quantization; it takes Cartesian products to tensor products.

iii) We have a canonical isomorphism  $\mathcal{M}_T(\bar{\Sigma}) \cong \overline{\mathcal{M}_T(\Sigma)}$ , where the bar on the right hand side means the dual symplectic manifold, i.e., equipped with minus the symplectic form. The isomorphism now follows from the fact that the opposite of an affine symplectic manifold is quantized by the dual Hilbert space.

iv) By definition (3.4), together with Theorem 2.18 and remark 2.20, we find:

$$\begin{aligned} E(\Sigma, \vec{\lambda}) &= \text{Hom}_{\widetilde{T(\partial\Sigma)}} \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}}, \mathcal{H}_{\Sigma} \right) \\ &\cong \bigoplus_{\vec{\lambda}_S \in H^1(S; \hat{A})} \text{Hom}_{\widetilde{T(\partial\Sigma_1 \sqcup \partial\Sigma_2)}} \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}} \otimes \mathcal{H}_{\vec{\lambda}_S} \otimes \mathcal{H}_{(\vec{\lambda}_S, \ell)}^*, \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2} \right) \\ &\cong \bigoplus_{\vec{\lambda}_S \in H^1(S; \hat{A})} E(\Sigma_1, \vec{\lambda}_1, \vec{\lambda}_S) \otimes E(\Sigma_2, \vec{\lambda}_S, \vec{\lambda}_2). \end{aligned}$$

This proves factorization.

v) By Theorem 3.2 we have

$$\begin{aligned} \text{Det}_{\Sigma}^c \otimes E(\Sigma, \vec{\lambda}) &= \text{Det}_{V(\Sigma)}^{1/2} \otimes \text{Hom}_{\widetilde{T(\partial\Sigma)}} \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}}, \mathcal{H}_{\Sigma} \right) \\ &= \text{Hom}_{\widetilde{T(\partial\Sigma)}} \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{(\vec{\lambda}_{\text{out}}, \ell)}, \mathcal{H}_{\Sigma} \otimes \text{Det}_{V(\Sigma)}^{-1/2} \right). \end{aligned}$$

It follows from Theorem 1.11 that the tensor product  $\mathcal{H}_{\Sigma} \otimes \text{Det}_{V(\Sigma)}^{-1/2}$  is canonically flat over  $\mathcal{E}(\Sigma)$ , as are the irreducible representations of  $T(\partial\Sigma)$  in an obvious manner. Therefore  $\text{Det}_{\Sigma}^c \otimes E(\Sigma, \vec{\lambda})$  is a flat hermitian holomorphic vector bundle.  $\square$

**Proposition 3.7.** *There is a canonical isometric embedding*

$$\text{Det}_{\Sigma}^c \otimes E(\Sigma, \vec{\lambda}) \hookrightarrow \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}},$$

which identifies  $E(\Sigma, \vec{\lambda})$  as a closed subspace of  $\mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}}$ .

*Proof.* Recall from Theorem 3.2 that the line bundle  $\text{Det}_{\Sigma}^c$  is canonically isomorphic to  $\text{Det}_{V(\Sigma)}^{1/2}$ , which in turn, by Theorem 1.11 and Proposition 2.13 equals the line bundle determined by the basepoint of  $\mathcal{M}_T(\Sigma)$  given by the trivial bundle. In other words, a point  $\alpha \in \text{Det}_{\Sigma}^{-c}$  uniquely determines a basevector  $\Omega_{\Sigma}^{\alpha} \in \mathcal{H}_{\Sigma}$ . Consider now the map

$$\alpha \otimes \psi \mapsto \langle \psi(\dots), \Omega_{\Sigma}^{\alpha} \rangle_{\mathcal{H}_{\Sigma}} \in \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}},$$

where  $\psi \in E(\Sigma, \vec{\lambda})$ . Notice that this formula a priori defines an element in conjugate dual of the Hilbert space the right hand side, so that it is essential that we work with the Hilbert space representations of the loop group. An easy computation, as in the proof of Lemma 2.19, shows that this map is isometric, i.e.,

$$\|\alpha \otimes \psi\|^2 = \left\| \langle \psi(\dots), \Omega_{\Sigma}^{\alpha} \rangle_{\mathcal{H}_{\Sigma}} \right\|^2.$$

It therefore must be an embedding onto a closed subspace of  $\mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}}$ . This proves the proposition.  $\square$



**3.4. Conformal blocks.** The result of this section is not needed elsewhere in the paper. Its purpose only serves to show how the standard approach to modular functors from representations of loop groups, cf. [S3], fits into our framework. The starting point for this is the embedding of Proposition 3.7.

**Theorem 3.8.** *The embedding of Proposition 3.7 defines a canonical isomorphism*

$$\text{Det}_\Sigma^\epsilon \otimes E(\Sigma, \vec{\lambda}) \cong \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}} \right)^{T_C^\Sigma}.$$

**Remark 3.9.** The complexification  $T_C(\partial\Sigma)$  of the loop group does not act on a positive energy representation by bounded operators, so part of the statement of the theorem is that  $T_C^\Sigma$  acts by exponentials of closed operators with a joint maximal domain. With this, the right hand side is a closed subspace of  $\mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}}$ .

*Proof.* This is proved by constructing an irreducible representation of  $\widetilde{H^1(\Sigma; A)}$  on the space of invariants on the right hand side. Let us, for notational simplicity only, consider the case when  $\Sigma$  has only outgoing boundary components. Consider now the closed subgroup  $T_C^\Sigma \subseteq T_C(\partial\Sigma)$ . It fits into an exact sequence

$$0 \rightarrow H^0(\Sigma, \Lambda) \rightarrow \mathfrak{t}_C^\Sigma \rightarrow T_C^\Sigma \rightarrow H^1(\Sigma, \Lambda) \rightarrow 1,$$

and therefore  $\pi_0(T_C^\Sigma) = H^1(\Sigma, \Lambda)$ . The natural map  $\pi_0(T_C^\Sigma) \rightarrow \pi_0(LT_C) = H^1(\partial\Sigma, \Lambda)$  induced by the inclusion  $T_C^\Sigma \subseteq LT_C$  is nothing but the restriction map  $i_{\partial\Sigma}^*$  to the boundary which fits into an exact sequence

$$(3.5) \quad H^1(\Sigma, \partial\Sigma; \Lambda) \longrightarrow H^1(\Sigma; \Lambda) \xrightarrow{i_{\partial\Sigma}^*} H^1(\partial\Sigma; \Lambda) \xrightarrow{\delta} H^2(\Sigma, \partial\Sigma; \Lambda) \rightarrow 0.$$

As above, we first consider  $H^0(\Sigma, T_C) \subseteq T_C^\Sigma$ . Taking invariants, we find

$$(3.6) \quad \mathcal{H}_{\vec{\lambda}_{\text{out}}}^{H^0(\Sigma, T_C)} = \bigoplus_{\substack{\vec{\mu} \in H^1(\partial\Sigma; \Lambda) \\ \delta(\vec{\lambda} + \vec{\mu}) = 0}} \mathcal{H}_{\vec{\lambda} + \vec{\mu}},$$

since  $\delta(\vec{\lambda} + \vec{\mu}) \in H^2(\Sigma, \partial\Sigma; \Lambda^\circ)$  is the character by which  $H^0(\Sigma, T_C)$  acts on the Hilbert space. This space carries a residual action of  $\Gamma_\Sigma := T_C^\Sigma / H^0(\Sigma, T_C)$ , a complex Lie group with Lie algebra

$$V_C^\Sigma = \mathfrak{t}_C^\Sigma / H^0(\Sigma, \mathfrak{t}_C) \subset V_C(\partial\Sigma).$$

The central extension of the Lie algebra on the right hand side is determined by the cocycle induced from (2.4). By Cauchy's theorem it is a complex isotropic subspace, and, even stronger, it is positive and compatible with the polarization  $V_C(\partial\Sigma) = V_C^+(\partial\Sigma) \oplus V_C^-(\partial\Sigma)$  in the sense of [S3, Def. 10.5]. This means that because a typical component  $\mathcal{H}_{\vec{\lambda} + \vec{\mu}}$  of (3.6) is simply the standard Bosonic Fock space of  $V_C(\partial\Sigma)$  associated to the polarization as in [S1, §3], we can think of  $V_C^\Sigma$  as acting by “lowering operators”. But, acting on the dense domain of finite energy vectors, these are pro-nilpotent and therefore closed. Because the Lie algebra is abelian, they have a common maximal domain and the subspace of vectors in  $\mathcal{H}_{(\vec{\lambda} + \vec{\mu})}$  annihilated by  $V_C^\Sigma$  is closed. It follows from the Bosonic commutator relations that

this space carries an irreducible representation of the induced Heisenberg extension of  $(V_{\mathbb{C}}^{\Sigma})^{\circ} / V_{\mathbb{C}}^{\Sigma}$ . By Cauchy's theorem, one deduces that

$$(V_{\mathbb{C}}^{\Sigma})^{\circ} \cong \{\alpha \in \Omega_{hol}^1(\Sigma, \mathfrak{t}_{\mathbb{C}}), d\alpha = 0, \alpha|_{\partial\Sigma} \text{ is exact}\},$$

the space of closed holomorphic differentials on  $\Sigma$ . Here, the isomorphism is given by the map  $\alpha \mapsto f \in V_{\mathbb{C}}(\partial\Sigma), \alpha|_{\partial\Sigma} = df$ . Since  $\Sigma$  is a Stein manifold, one finds

$$\begin{aligned} (V_{\mathbb{C}}^{\Sigma})^{\circ} / V_{\mathbb{C}}^{\Sigma} &\cong \{\alpha \in \Omega_{hol}^1(\Sigma, \mathfrak{t}_{\mathbb{C}}), d\alpha = 0, \alpha|_{\partial\Sigma} \text{ is exact}\} / \{d\beta, \beta \in \Omega_{hol}^0(\Sigma, \mathfrak{t}_{\mathbb{C}})\} \\ &\cong \ker \left( H^1(\Sigma; \mathfrak{t}_{\mathbb{C}}) \rightarrow H^1(\partial\Sigma, \mathfrak{t}_{\mathbb{C}}) \right), \\ &\cong H^1(\hat{\Sigma}, \mathfrak{t}_{\mathbb{C}}), \end{aligned}$$

with the induced symplectic form equal to cup product combined with the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{t}_{\mathbb{C}}$  and evaluation on the fundamental class. The Heisenberg extension of the real form  $H^1(\hat{\Sigma}, \mathfrak{t})$  will act unitarily.

This takes care of the identity component of  $T_{\mathbb{C}}^{\Sigma}$ , and what remains is the action of the group of components  $\pi_0(T_{\mathbb{C}}^{\Sigma}) = H^1(\Sigma; \Lambda)$ . This acts in a two-fold way as a positive complex isotropic lattice. First of all, its image in  $H^1(\partial\Sigma; \Lambda)$  under  $i_{\partial\Sigma}^*$  acts on the Hilbert space (3.6) by maps  $L_{\vec{\mu}} : \mathcal{H}_{\vec{\lambda}} \rightarrow \mathcal{H}_{\vec{\lambda}+\vec{\mu}}$  given by

$$L_{\vec{\mu}}(\psi_{\vec{\lambda}}) = e^{-\langle \vec{\mu}, \vec{\lambda} \rangle - \langle \vec{\mu}, \vec{\mu} \rangle / 2} \psi_{\vec{\lambda}+\vec{\mu}},$$

where  $\psi_{\vec{\lambda}} \in \mathcal{H}_{\vec{\lambda}}$  and  $\psi_{\vec{\lambda}+\vec{\mu}}$  is simply the same vector now considered as an element of  $\mathcal{H}_{\vec{\lambda}+\vec{\mu}}$ . The exponential pre-factor appears because of the fact the lattice is positive and complex isotropic. By exactness of the sequence (3.5), that is,  $\text{im}(i_{\partial\Sigma}^*) = \ker(\delta)$  and the invariant vectors are all Gaussian-shaped of the form

$$\sum_{\vec{\mu}} e^{-\langle \vec{\mu}, \vec{\mu} \rangle / 2} \psi_{\vec{\lambda}+\vec{\mu}},$$

and completely determined by  $\psi_{\vec{\lambda}} \in \mathcal{H}_{\vec{\lambda}}$ . Therefore this part of the lattice  $H^1(\Sigma; \Lambda)$  kills the direct sum in (3.6). What is left is a canonical representation of  $H^0(\partial\Sigma; A)$  by the character given by  $\vec{\lambda}$ .

Finally, consider the kernel of  $i_{\partial\Sigma}^*$ . Again by exactness, this is equal to image of  $H^1(\Sigma, \partial\Sigma; \Lambda)$  which embeds into  $H^1(\hat{\Sigma}, \mathfrak{t}_{\mathbb{C}})$  as a complex isotropic positive lattice that can be identified as  $H^1(\hat{\Sigma}; i\Lambda)$ . As above, one proves that the invariant subspace is an irreducible representation of

$$H^1(\hat{\Sigma}; \Lambda^{\circ}) / H^1(\hat{\Sigma}; \Lambda) = H^1(\hat{\Sigma}; A).$$

By Proposition 3.1, the invariant subspace therefore carries an irreducible representation of the Heisenberg group of  $H_1(\Sigma; A_{\ell})$  with center  $H_1(\partial\Sigma; A)$  acting according to  $\vec{\lambda} \in H^0(\partial\Sigma; \hat{A})$ . One checks that the embedding of Proposition 3.7 is equivariant. It therefore must be an isomorphism.  $\square$

**3.5. The nonlinear  $\sigma$ -model.** As explained in [S3], the existence of a conformal field theory follows from the unitarity of the associated modular functor. In this case it is called the nonlinear  $\sigma$ -model, which describes strings moving on the Riemannian manifold  $T$ . It is an abelian version of the so-called WZW-model, which describes strings moving on an arbitrary compact Lie group.

**Definition 3.10** (cf. [S3]). A unitary conformal field theory is given by a smooth projective monoidal  $*$ -functor  $\Psi$  from the complex cobordism category  $\mathcal{E}$  to the category  $Hilb$  of complex Hilbert spaces and trace class maps.

Notice that the “category” of Hilbert spaces and trace class operators fails to be a true category for the same reason as  $\mathcal{E}$ : the identity mapping on an infinite dimensional Hilbert space is not trace class. Below, we will explicitly spell out the details of this definition.

To define a CFT, i.e., a functor  $\Psi : \hat{\mathcal{E}} \rightarrow Hilb$ , one needs the following data:

- i) a Hilbert space  $\mathcal{H}_{S^1}$ , such that  $\Psi(C_n) = \mathcal{H}_{S^1}^{\otimes n}$ , i.e.,  $\Psi$  is monoidal,
- ii) a trace class operator  $\Psi_{(\Sigma, \alpha)} : \mathcal{H}_{in} \rightarrow \mathcal{H}_{out}$ , for each Riemann surface  $\Sigma$  and

$$\alpha \in L_{\Sigma}^{(p,q)} := \text{Det}_{\Sigma}^{\otimes p} \otimes \overline{\text{Det}_{\Sigma}^{\otimes q}}, \quad p, q \in \mathbb{C}$$

which only depends on the conformal equivalence class of  $\Sigma$ ,

subject to the conditions:

- The equality

$$\Psi_{(\Sigma, \alpha)} = \Psi_{(\Sigma_1, \alpha_1)} \circ \Psi_{(\Sigma_2, \alpha_2)},$$

whenever  $\Sigma = \Sigma_1 \cup_{C_k} \Sigma_2$  and  $\alpha$  equals the image of  $\alpha_1 \otimes \alpha_2$  under the factorization isomorphism

$$L_{\Sigma_1}^{(p,q)} \otimes L_{\Sigma_2}^{(p,q)} \xrightarrow{\cong} L_{\Sigma}^{(p,q)}.$$

- $\Psi_{(\bar{\Sigma}, \bar{\alpha})} = \Psi_{(\Sigma, \alpha)}^*$ , i.e.,  $\Psi$  is a  $*$ -functor.

A conformal field theory for which  $q = 0$  is called *chiral*. The pair  $(p, q)$  is referred to as the *central charge*. Below we will have  $p = q$  and we use this terminology for this single element of  $\mathbb{C}$ .

In our case, the nonlinear  $\sigma$ -model, we use the theory of positive energy representations of  $LT$  to define the basic Hilbert space. For a fixed level  $\ell \in \mathbb{N}$  define

$$(3.7) \quad \mathcal{H}_{S^1} = \bigoplus_{\lambda \in \hat{A}} \mathcal{H}_{\varphi} \otimes \mathcal{H}_{\varphi}^*.$$

By the monoidal property, this defines the Hilbert space associated to  $C_n$  for all  $n \in \mathbb{N}$ . Next, we construct a trace class operator  $\Psi_{(\Sigma, \alpha)} : \mathcal{H}_{in} \rightarrow \mathcal{H}_{out}$ , where  $\mathcal{H}_{in}, \mathcal{H}_{out}$  are appropriate tensor products of  $\mathcal{H}_{S^1}$  corresponding to  $\partial\Sigma$ , and  $\Sigma$  is a Riemann surface with incoming and outgoing boundaries and  $\alpha \in L_{\Sigma}^{(c,c)}$  with central charge given by  $c = \ell \dim \mathfrak{t}$ .

For a labeling  $\vec{\lambda} = (\vec{\lambda}_{in}, \vec{\lambda}_{out})$  of the boundaries of  $\Sigma$ , we have the finite dimensional Hilbert space  $E(\Sigma, \vec{\lambda})$ , depending only on the conformal equivalence class of  $\Sigma$ , i.e., its image in  $\mathcal{E}$ . Since  $E(\Sigma, \vec{\lambda})$  is finite dimensional the inner product defines a vector

$$\Psi_{\Sigma, \vec{\lambda}}^{\langle \cdot, \cdot \rangle} \in E(\Sigma, \vec{\lambda}) \otimes \overline{E(\Sigma, \vec{\lambda})}.$$

By Proposition 3.7, for  $\alpha \in L_\Sigma^{(c,c)}$  we therefore find a vector

$$\begin{aligned} \Psi_{\Sigma, \vec{\lambda}}^\alpha &:= \alpha \otimes \Psi_{\Sigma, \vec{\lambda}}^{\langle \cdot, \cdot \rangle} \in \text{Det}_\Sigma^c \otimes E(\Sigma, \vec{\lambda}) \otimes \overline{\text{Det}_\Sigma^c} \otimes \overline{E(\Sigma, \vec{\lambda})} \\ &\subseteq \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}} \right) \otimes \overline{\left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}} \right)} \\ &\subseteq \text{Hom} \left( \mathcal{H}_{\vec{\lambda}_{\text{in}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{in}}}, \mathcal{H}_{\vec{\lambda}_{\text{out}}}^* \otimes \mathcal{H}_{\vec{\lambda}_{\text{out}}} \right) \end{aligned}$$

which is clearly trace-class. With this, we put

$$(3.8) \quad \Psi_{(\Sigma, \alpha)} := \bigoplus_{\vec{\lambda} \in H^1(\partial \Sigma; \hat{A})} \Psi_{\Sigma, \vec{\lambda}}^\alpha : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{out}},$$

a trace class operator between tensor products of the Hilbert space (3.7). We now arrive at the main conclusion:

**Theorem 3.11** (“Nonlinear  $\sigma$ -model”). *The Hilbert space  $\mathcal{H}_{S^1}$  and the operators  $\Psi_{(\Sigma, \alpha)}$  constitute a conformal field theory with central charge  $c = \ell \dim(\mathfrak{t})$ .*

*Proof.* Most importantly, we have to check the composition property, i.e., that  $\Psi$  indeed defines a functor from a central extension of  $\mathcal{E}$  to *Hilb*. By Theorem 3.6, the map  $(\Sigma, \vec{\lambda}) \mapsto E(\Sigma, \vec{\lambda})$  forms a unitary modular functor, and therefore the canonical morphism

$$\bigoplus_{\vec{\lambda}_S \in H^1(S, \hat{A})} E(\Sigma_1, \sqcup \Sigma_2, \vec{\lambda}, \vec{\lambda}_S, \vec{\lambda}_S) \otimes E(\Sigma_1, \sqcup \Sigma_2, \vec{\lambda}, \vec{\lambda}_S, \vec{\lambda}_S) \xrightarrow{\cong} E(\Sigma, \vec{\lambda}) \otimes E(\Sigma, \vec{\lambda})$$

induced by factorization will map the vector

$$\bigoplus_{\vec{\lambda}_S \in H^1(S, \hat{A})} \Psi_{\Sigma, \vec{\lambda}, \vec{\lambda}_S, \vec{\lambda}_S}^{\langle \cdot, \cdot \rangle}$$

to the vector  $\Psi_{\Sigma, \vec{\lambda}}^{\langle \cdot, \cdot \rangle} \in E(\Sigma, \vec{\lambda}) \otimes E(\Sigma, \vec{\lambda})$ . Combined with the determinant line bundle, which satisfies the gluing properties of Theorem 3.2, this means that that

$$\Psi_{\Sigma_1} \circ \Psi_{\Sigma_2} = \Psi_{\Sigma},$$

up to an element in  $\mathbb{C}$ , where  $\Sigma = \Sigma_1 \cup_S \Sigma_2$ . This proves functoriality of  $\Psi$ . It is a  $*$ -functor,  $\Psi_{(\Sigma, \alpha)}^* = \Psi_{(\bar{\Sigma}, \bar{\alpha})}$  because of property *iii*) in Theorem 3.6.  $\square$

**Remark 3.12.** Again, for odd levels, the construction uses a choice of spin structure. One therefore finds a so-called *spin*-conformal field theory. In this case, it is the 2-dimensional part of the theory described in [BM].

The preceding theorem does not a priori refer to the representation of the loop  $LT$  on  $\mathcal{H}_{S^1}$ , although of course its construction depends heavily on this structure. There is a way to “gauge” this symmetry, which leads to the following structure: in Proposition 3.7, we used the basepoint of  $\mathcal{M}_T(\Sigma)$ , i.e., the trivial bundle, and the determinant line  $\text{Det}_\Sigma^c$  to construct the embedding which lead to the trace class operator of Theorem 3.11. Instead, we can use any other point in  $\mathcal{M}_T(\Sigma)$ , i.e., any other holomorphic  $T_{\mathbb{C}}$ -bundle  $E$ , and the line  $\text{Det}_{\Sigma, E}^c$  associated to  $E$  via the embedding of Proposition 2.13. This leads to the following:

**Theorem 3.13** (“Gauged nonlinear  $\sigma$ -model”). *For any holomorphic  $T_{\mathbb{C}}$ -bundle  $E$  over  $\Sigma$  and  $\alpha \in \overline{\text{Det}}_{\Sigma,E}^{\mathbb{C}} \otimes \text{Det}_{\Sigma,E}^{\mathbb{C}}$ , there is a unique trace class operator*

$$\Psi_{\Sigma,E}^{\alpha} : \mathcal{H}_{\text{in}} \rightarrow \mathcal{H}_{\text{out}},$$

*satisfying composition rules when composing complex cobordisms with holomorphic  $T_{\mathbb{C}}$ -bundles.*

**3.6. The category of Positive energy representations of  $LT$ .** Recall the definition 2.1 of a positive energy representations of  $LT$ . In the following, we will restrict to representations with the following property: when decomposed into irreducibles, each of the multiplicity spaces is required to be finite dimensional. Let  $\mathcal{C}$  denote the category of such representations at level  $\ell$ , with bounded intertwiners as morphisms. By the assumption above this category is abelian, and clearly semi-simple. Consider the bifunctor  $\langle \cdot, \cdot \rangle : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \underline{\text{Hilb}}$ , defined by

$$\langle \mathcal{H}_1, \mathcal{H}_2 \rangle := \text{Hom}_{\widetilde{LT} \times \widetilde{LT}_{\text{op}}} \left( \text{Ind}_{LT}^{\widetilde{LT} \times \widetilde{LT}_{\text{op}}}(\mathbb{C}), \mathcal{H}_1^* \otimes \mathcal{H}_2 \right),$$

where  $\underline{\text{Hilb}}$  is the category of finite dimensional Hilbert spaces. Thinking of  $\langle \cdot, \cdot \rangle$  as an “inner product”, it furnishes the category  $\mathcal{C}$  with the structure of a 2-Hilbert space [Ba]. Most important for us, any additive functor between 2-Hilbert spaces has an adjoint, unique up to natural isomorphism, determined by the usual formula familiar from the adjoint of an operator on a Hilbert space.

For a compact 1-manifold  $S$  let  $\mathcal{C}_S$  be the category of positive energy representations of  $T(S)$  with the above property of having finite multiplicity, so that  $\mathcal{C}_{S^1} = \mathcal{C}$ . For a complex cobordism  $\Sigma$ , abbreviate  $\mathcal{C}_{\partial\Sigma_{\text{in}}}, \mathcal{C}_{\partial\Sigma_{\text{out}}}$  by  $\mathcal{C}_{\text{in}}, \mathcal{C}_{\text{out}}$ . The fundamental structure theorem for the category  $\mathcal{C}$  is given as follows:

**Theorem 3.14.** *Any cobordism  $\Sigma$  induces a functor*

$$U_{\Sigma} : \mathcal{C}_{\text{in}} \rightarrow \mathcal{C}_{\text{out}},$$

*satisfying the following properties:*

i) *when  $\Sigma = \Sigma_1 \cup_{\mathbb{C}} \Sigma_2$ , there is a natural transformation*

$$U_{\Sigma} \cong U_{\Sigma_2} \circ U_{\Sigma_1},$$

ii)  *$U_A \cong \text{id}_{\mathcal{C}}$  for any  $A$  with the topology of a cylinder,*

iii) *there is a natural isomorphism*

$$U_{\Sigma}^* \cong U_{\overline{\Sigma}},$$

iv) *the group  $\text{Diff}^+(\Sigma)$  acts on  $U_{\Sigma}$  by natural transformations, and this action factors over the identity component of  $\text{Diff}^+(\Sigma, \partial\Sigma)$ ,*

v) *a complex structure on  $\Sigma$ , together with an element  $\alpha \in \text{Det}_{\Sigma}^{\mathbb{C}}$ , defines a map  $\psi_{\Sigma,\alpha} : E \rightarrow U_{\Sigma}(E)$ , for all objects  $\mathcal{H}$  of  $\mathcal{C}_{\text{in}}$ , such that*

$$\psi_{\Sigma,\alpha} = \psi_{\Sigma_2,\alpha_2} \circ \psi_{\Sigma_1,\alpha_1},$$

*under the natural transformation in i).*

Notice that the structure outlined in the theorem is close to what is called a “category valued topological quantum field theory” in [S2]. In the theorem above, we have ignored all structure coming from 3-dimensional topology. However, the structure above suffices to prove, as outlined in [S2], the following:

**Corollary 3.15.** *The category  $\mathcal{C}$  of positive energy representation of LT at level  $\ell$  with finite dimensional multiplicity spaces is a modular tensor category.*

## APPENDIX A. GENERALIZED HEISENBERG GROUPS

**A.1. Definition.** Following [FMS], we will define a certain class of infinite dimensional groups called *generalized Heisenberg groups*. Let  $A$  be an infinite dimensional abelian Lie group such that  $\text{Lie}(A)$  is a complete locally convex nuclear topological vector space and  $\pi_0(A)$  and  $\pi_1(A)$  are finitely generated discrete abelian groups. We assume that there is an exact sequence

$$(A.1) \quad 0 \rightarrow \pi_1(A) \rightarrow \text{Lie}(A) \xrightarrow{\exp} A \rightarrow \pi_0(A) \rightarrow 0,$$

which makes  $\text{Lie}(A)$  into a covering space of  $A$ , i.e., the exponential map is a local diffeomorphism. It follows that any such group can be written (non-canonically) as  $A \cong V \times T \times \pi$ , where  $V$  is a vector space,  $T$  a torus and  $\pi$  a discrete group. Let  $\psi$  be a group 2-cocycle, i.e., a smooth map  $\psi : A \times A \rightarrow \mathbb{T}$  satisfying

$$\psi(a_1, a_2)\psi(a_1 + a_2, a_3) = \psi(a_1, a_2 + a_3)\psi(a_2, a_3).$$

Consider the central extension

$$1 \rightarrow \mathbb{T} \rightarrow \tilde{A} \rightarrow A \rightarrow 0,$$

defined as  $\tilde{A} = A \times \mathbb{T}$  with multiplication

$$(a_1, z_1) \cdot (a_2, z_2) = (a_1 a_2, \psi(a_1, a_2) z_1 z_2).$$

The isomorphism class of this extension is determined by the commutator map  $s : A \times A \rightarrow \mathbb{T}$  defined by  $s(a_1, a_2) := \psi(a_1, a_2)\psi(a_2, a_1)^{-1}$ . We define  $Z(A) := \ker(s)$ , so that

$$\text{center}(\tilde{A}) = \widetilde{Z(A)}.$$

It was proved in [FMS] that any central extension of  $A$  is of this kind, i.e., topologically trivial and defined by a group cocycle  $\psi$ . The opposite of  $\tilde{A}$ , denoted  $\tilde{A}_{op}$ , is the central extension associated to the cocycle  $\psi^{-1}$ .

**Definition A.1.** The extension  $\tilde{A}$  is called a Heisenberg group when the commutator pairing is nondegenerate, i.e.,  $Z(A) = \{e\}$ . It is called a generalized Heisenberg group when its center  $Z(A)$  is a locally compact abelian group.

**A.2. The linear case: Gaussian measures.** As a special case, consider a symplectic vector space  $(V, \omega)$ . The symplectic form  $\omega$  defines a cocycle  $\psi = \exp i\pi\omega$  on the abelian group  $V$  and the associated central extension  $\tilde{V}$  as above is what is traditionally called the Heisenberg group. Of course this case is well-known, cf. [S1], but we review the theory for completeness. To construct irreducible representations of this Heisenberg group one needs an extra piece of structure on  $V$ , called a polarization:

**Definition A.2.** A *polarization* of a symplectic vector space is a compatible positive complex structure. Two complex structures  $J_1$  and  $J_2$  belong to the same polarization class if and only if  $J_1 - J_2$  is a Hilbert–Schmidt operator.

Of course, a complex structure on  $V$  is given by an endomorphism  $J : V \rightarrow V$  such that  $J^2 = -1$ . Such a complex structure is called

- *compatible with  $\omega$*  if  $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ , for all  $v_1, v_2 \in V$ ,

- *positive* if  $\omega(Jv, v) > 0$ , for all  $v \neq 0$ .

A specific choice of complex structure  $J$  belonging to a polarization class induces a hermitian pre-inner product

$$(A.2) \quad \langle v_1, v_2 \rangle = \omega(Jv_1, v_2) + i\omega(v_1, v_2).$$

The first term defines a real metric on  $V$  that we denote by  $Q$ . Since, by assumption,  $V$  is nuclear,  $Q$  determines a family  $\mu_J^t$ ,  $t > 0$  of Gaussian measures on the dual  $V^*$ , uniquely determined by its Fourier transform

$$\int_{V^*} e^{i\tilde{\xi}(v)} d\mu_J^t(\tilde{\xi}) = e^{-tQ(v,v)/2},$$

for all  $v \in V$ , cf. [GV, chapter IV]. This measure is not  $V$ -invariant. Rather, its quasi-invariance under the translation action of  $V$  is given by the Cameron–Martin formula for the Radon–Nikodym derivative:

$$(A.3) \quad \frac{d\mu_J^t(\tilde{\xi} + v)}{d\mu_J^t(\tilde{\xi})} = e^{-(\frac{1}{2}Q(v,v) + Q(v,\tilde{\xi}))/t},$$

for all  $v \in V$ . Let  $L_{hol}^2(V^*, d\mu)$  be the Hilbert space of square integrable holomorphic functions on  $V^*$  with respect to the measure  $\mu := \mu^1$ , with inner product defined by

$$\langle f_1, f_2 \rangle = \int_{V^*} \bar{f}_1(\tilde{\xi}) f_2(\tilde{\xi}) d\mu(\tilde{\xi}).$$

Let  $V$  act on  $f \in L_{hol}^2(V^*, d\mu)$  by

$$(v \cdot f)(\tilde{\xi}) := e^{-(\langle v, \tilde{\xi} \rangle / 2 + \langle v, v \rangle / 4)} f(\tilde{\xi} + v).$$

A direct computation using the Cameron–Martin formula (A.3) shows that this action is unitary, i.e.,  $\langle v \cdot f_1, v \cdot f_2 \rangle = \langle f_1, f_2 \rangle$  and defines a representation of the Heisenberg group of  $V$ . This representation is irreducible and denoted by  $\mathcal{H}_V$ , or perhaps  $\mathcal{H}_{V_J}$  if we want to stipulate the dependence on the chosen complex structure  $J$ , such as in the following.

**A.2.1. Shale’s theorem.** So far, the irreducible representation  $\mathcal{H}_V$  of  $\tilde{V}$  depends on the specific choice of a compatible, positive complex structure  $J$  on  $V$ . However,  $\mathcal{H}_V$  turns out only to depend on the polarization class determined by  $J$ . Introduce the infinite dimensional “Siegel upper half space”

$$\mathcal{J}(V) = \left\{ \begin{array}{c} \text{Positive compatible} \\ \text{complex structures } J' \text{ on } (V, \omega) \end{array} \middle| J - J' \text{ is Hilbert–Schmidt} \right\}.$$

There are other descriptions of this space: alternatively we have

$$\mathcal{J}(V) = \left\{ \begin{array}{c} \text{Hilbert schmidt operators } T : A \rightarrow \bar{A} \\ \text{ii) } 1 - TT^* > 0. \end{array} \middle| \begin{array}{c} \text{i) } \omega(Tv_1, v_2) = \omega(Tv_2, v_1) \end{array} \right\}.$$

Finally, introduce the *restricted symplectic group*  $Sp_{res}(V)$  of the polarized symplectic vector space  $V$  by

$$Sp_{res}(V) := \{A \in Sp(V), [A, J] \text{ is Hilbert–Schmidt}\},$$

for any  $J$  belonging to the polarization class. This group acts transitively on  $\mathcal{J}(V)$  and one has

$$\mathcal{J}(V) = Sp_{res}(V) / U(V_J),$$

where  $V_J$  means the complex pre-Hilbert space formed by  $J$  by means of the pre-inner product (A.2). We now have:

**Theorem A.3** (Shale [Sh]). *Two representations  $\mathcal{H}_{V_{J_1}}$  and  $\mathcal{H}_{V_{J_2}}$  are equivalent if and only if  $J_1 - J_2$  is Hilbert–schmidt, i.e., if  $J_1$  and  $J_2$  belong to the same polarization class.*

In terms of Gaussian measures, this can be derived from the following: Let  $J_T \in \mathcal{J}(V)$  be another complex structure given in terms of a Hilbert–Schmidt map  $T : \tilde{A} \rightarrow A$ . The Gaussian measure  $\mu_{J_T}$  is absolutely continuous with respect to  $\mu_J$  with Radon–Nikodym derivative

$$(A.4) \quad \frac{d\mu_{J_T}(\xi)}{d\mu_J(\xi)} = \det(1 - T^*T)^{-1/2} \exp\left(-\frac{1}{2} \langle T\xi, T\xi \rangle\right),$$

cf. [BSZ, Thm. 4.5]. In this formula,  $\det(\dots)$  is the Fredholm determinant of an operator of the form  $1 + \text{trace class}$ .

By Schur’s lemma, the intertwiner establishing the equivalence of two representations is unique up to a scalar. Consequently, the Hilbert space  $\mathcal{H}_V$  carries a projective unitary representation of  $Sp_{res}(V)$  which extends the representation of  $\tilde{V}$  to the semi-direct product  $Sp_{res}(V) \ltimes V$ . This is the infinite dimensional analogue of the metaplectic representation.

Shale’s theorem implies that the bundle of projective spaces  $\mathbb{P}\mathcal{H}_V$  is canonically flat over  $\mathcal{J}(V)$ , in other words the bundle of Hilbert spaces  $\mathcal{H}_{V_J}$  for  $J \in \mathcal{J}(V)$  is projectively flat. Its failure to be truly flat, i.e., not projectively, is measured by the line bundle  $\text{Det}_V^{1/2}$  over  $\mathcal{J}(V)$  whose fiber at  $J \in \mathcal{J}(V)$  is the ray in  $\mathcal{H}_{V_J}$  spanned by the lowest weight vector  $\Omega_J \in \mathcal{H}_{V_J}$ , cf. Proposition 1.10. This line bundle is holomorphic and hermitian and earns its notation from the fact that

$$\text{Det}_V^{1/2} \otimes \text{Det}_V^{1/2} \cong \text{Det}_V,$$

where  $\text{Det}_V$  is the line bundle over  $\mathcal{J}(V)$  defined as the determinant of the family of index zero Fredholm operators  $F_{J'} = (1 - JJ')/2$  parameterised by  $J' \in \mathcal{J}(V)$ . Therefore we see that the bundle

$$\text{Det}_V^{-1/2} \otimes \mathcal{H}_V,$$

where  $\text{Det}_V^{-1/2}$  is the dual of  $\text{Det}_V^{1/2}$ , carries a canonical fiber-wise inner product, turning it into a flat bundle of Hilbert spaces over  $\mathcal{J}(V)$ .

**A.3. Polarizations and irreducible representations.** Returning to the general case of a generalized Heisenberg group  $\tilde{A}$ , notice that the defining cocycle induces a skew symmetric bilinear form

$$S : \text{Lie}(A) \times \text{Lie}(A) \rightarrow \mathbb{R},$$

with finite dimensional kernel  $\ker(S) \subseteq \text{Lie}(A)$ . The quotient vector space

$$V(A) := \text{Lie}(A) / \ker(S)$$

is therefore symplectic and in general infinite dimensional. To discuss the representation theory, we need an appropriate generalization of the notion of polarization. Although there seem to be many variants of this, all more or less equivalent, for us the following definition is very useful:

**Definition A.4** (compare [S3, Def. 10.3]). A *polarization* of  $A$  means an operator  $J : \text{Lie}(A) \rightarrow \text{Lie}(A)$ , satisfying



- i)  $S(J\zeta, J\eta) = S(\zeta, \eta)$  for all  $\zeta, \eta \in \text{Lie}(A)$ ,
- ii)  $J$  induces a compatible positive complex structure on  $V(A)$ .

Two polarizations define the same *polarization class* if they induce the same polarization class for the symplectic vector space  $V(A)$ .

Notice that to define the polarization class on  $A$ , it suffices to give the operator  $J$  up to Hilbert–Schmidt operators. The classification theorem of irreducible representations of polarized generalized Heisenberg groups is now as follows:

**Theorem A.5.** [FMS] *A polarization class and a splitting  $\chi : Z(A) \rightarrow \mathbb{T}$  of the central extension*

$$1 \rightarrow \mathbb{T} \rightarrow \widetilde{Z(A)} \rightarrow Z(A) \rightarrow 1,$$

*determine a unique irreducible representation of  $\tilde{A}$  in which the center acts according to  $\chi$ . For a given polarization class, this defines a bijective correspondence between such splittings and irreducible representations.*

Notice that the theorem implies that up to isomorphism, the irreducible representation  $\mathcal{H}_{A, \chi}$ , for fixed character  $\chi$ , only depends on the polarization class, not just the specific polarization. The theorem above unifies Shale’s theorem with Mackey’s version of the Stone–von Neumann theorem for locally compact abelian groups –satisfying the assumptions stated in §A.1– together with the Fourier transform for abelian groups. By the spectral theorem, or equivalently, the Fourier transform over the center, any unitary representation of  $\tilde{A}$  on  $\mathcal{H}$  decomposes as

$$\mathcal{H} \cong \int_{\widetilde{Z(A)}}^{\oplus} \mathcal{H}_{\chi} d\mu(\chi),$$

where the spectral measure  $\mu$  on  $\widetilde{Z(A)}$  is equivalent to the Haar measure when  $Z(A)$  is locally compact. It follows that in an irreducible representation  $Z(A)$  must act by a fixed character.

**Example A.6.** A good example is provided by the level  $\ell$  central extension of the loop group  $LT$ . As shown in §2.2 this is a generalized Heisenberg group with center  $\mathbb{T} \times A_{\ell}$ . In this case, there is a natural polarization determined by the canonical automorphic circle action: The circle group naturally acts by rotations of the circle, and the induced action on the Lie algebra  $Lt$  defines a polarization: Indeed the induced action on the associated symplectic vector space  $V(S^1) = Lt/\mathfrak{t}$  equipped with the symplectic form induced from the cocycle (2.4) is generated by the Energy functional

$$E(\xi) = \omega \left( \frac{d}{d\theta} \xi, \xi \right) = \int_0^{2\pi} \left\langle \frac{d\xi}{d\theta}, \frac{d\xi}{d\theta} \right\rangle d\theta,$$

for  $\xi \in Lt$ . This is compatible with a unique complex structure essentially given by the Hilbert transform. This is equivalent to the decomposition  $V_{\mathbb{C}}(S^1) = V_+(S^1) \oplus V_-(S^1)$  into positive, resp. negative Fourier coefficients. With this, the class of representations for this polarization class of  $LT$  are exactly the positive energy representations in the sense of Definition 2.1, and the irreducible representations constructed in §2.2 give an explicit realization of the classification of Theorem A.5.

**A.4. Induction from isotropic subgroups.** Let  $(A, \psi)$  be a generalized Heisenberg group and denote by  $\hat{A}$  the dual group of continuous characters. By Definition A.1, a Heisenberg group comes equipped with a map

$$e : A \rightarrow \hat{A},$$

defined as  $e(a_1)(a_2) := s(a_1, a_2)$ , whose kernel equals  $Z(A)$ . When  $Z(A) = \{e\}$ , i.e.,  $(A, \psi)$  is a Heisenberg group, its image is dense in  $\hat{A}$ .

Let  $B \subset A$  be isotropic. This means that one of the following equivalent conditions is satisfied:

- the extension  $\tilde{B}$  induced by  $\tilde{A}$  is abelian,
- $s|_{B \times B} = 1$ ,
- $B \subseteq B^\perp$ , where  $B^\perp := \{a \in A, s(a, b) = 1, \text{ for all } b \in B\}$ .

When  $B$  is isotropic and maximal with respect to any of the above properties, it is said to be Lagrangian. Because the induced central extension  $\tilde{B}$  is abelian, it must be trivial, however not necessarily in a canonical way. A trivialization is determined by a splitting, i.e., a map  $\chi : B \rightarrow \mathbb{T}$  satisfying

$$\chi(b_1 + b_2) = \chi(b_1)\chi(b_2)\psi(b_1, b_2),$$

for all  $b_1, b_2 \in B$ . As such it determines a lifting  $B \hookrightarrow \tilde{B}$  by  $b \mapsto (b, \chi(b))$ . Alternatively, such a splitting can be viewed as a one-dimensional unitary representation of  $\tilde{B}$ . Via the splitting  $\chi$  over  $B$ , we can restrict unitary representations of  $\tilde{A}$  to  $B$ .

**Proposition A.7 (Spectral Theorem).** *Let  $U$  be a unitary representation of  $\tilde{A}$  on  $\mathcal{H}$ . There exists a measure  $\mu^\chi$  on  $\hat{B}$ , unique up to equivalence and quasi-invariant under  $A$ , which disintegrates  $\mathcal{H}$ ;*

$$\mathcal{H} \cong \int_{\hat{B}}^{\oplus} \mathcal{H}_\xi d\mu^\chi(\xi)$$

so that each  $\mathcal{H}_\xi$ ,  $\xi \in \hat{B}$ , carries a unitary representation of  $\tilde{B}^\perp$  with  $B \subset \text{center}(\tilde{B}^\perp)$  acting via  $\xi$ . In fact:

$$(\mathcal{H}, U) \text{ irreducible} \iff \mathcal{H}_\xi \text{ irreducible for } \tilde{B}^\perp \text{ almost everywhere.}$$

*Proof.* Existence and uniqueness of the measure is given by the spectral theorem applied to the commutative operator algebra generated by the action of  $B$  on  $\mathcal{H}$ . With this disintegration,  $B$  acts by

$$(U(b)f)(\xi) = \xi(b)f(\xi),$$

where  $\xi \in \hat{B}$  and  $b \in B$ . For  $a \in A$ , we have, by definition

$$U(a)U(b) = \psi(a, b)U(a + b),$$

and therefore conjugation by  $U(a)$  in  $\mathcal{H}$  preserves the operator algebras generated by  $B$  and  $A$ . Therefore  $U(a)$  acts on  $\mathcal{H}$  with respect to the disintegration as

$$(U(a)f)(\xi) = \sqrt{\frac{d\mu(\xi + e(a))}{d\mu(\xi)}} U_\xi(a)(f(\xi)),$$

where  $U_\xi(a) : \mathcal{H}_\xi \rightarrow \mathcal{H}_{\xi + e(a)}$ ,  $\xi \in \hat{B}$  is a measurable family of unitary operators. In the formula above, the expression in the brackets is the Radon–Nikodym derivative of the  $e(a)$ -translate of the measure  $\mu$  with respect to itself. Restricted to  $B^\perp \subseteq A$ , this translation action on  $\hat{B}$ -action is trivial, so we see that  $\mathcal{H}_\xi$  carries

a unitary representation of  $\tilde{B}^\perp$  for each  $\xi \in \hat{B}$ . The last statement now follows easily.  $\square$

If  $B$  is Lagrangian,  $\tilde{B}^\perp = \tilde{B}$  is abelian and therefore  $\mathcal{H}_\xi \cong \mathbb{C}$  when  $\mathcal{H}$  is irreducible, so we find that  $\mathcal{H} \cong L^2(\hat{B}, d\mu)$ . In fact, in this case we can find an explicit expression for the measure, not just its class, by choosing a cyclic vector  $\Omega \in \mathcal{H}$  for the action of  $B$ . This determines a positive definite function on  $B$  by  $b \mapsto \langle \Omega, U(b)\Omega \rangle$ . By the Bochner theorem for nuclear abelian Lie groups, cf. [B], it is the Fourier transform of a unique Borel measure  $\mu_\Omega$ :

$$\langle \Omega, U(b)\Omega \rangle = \int_{\hat{B}} \xi(b) d\mu_\Omega(\xi),$$

for all  $b \in B$ . With this measure, the isomorphism  $\mathcal{H} \cong L^2(\hat{B}, d\mu_\Omega)$  is given by continuous extension of the map  $U(b)\Omega \mapsto \hat{b}$ , where  $\hat{b} \in L^2(\hat{B}, d\mu_\Omega)$  is defined as  $\hat{b}(\xi) := \xi(b)$ . The argument is the same as in [GV, Chapter IV, §5.4].

With these preparations, we can now define induction. We do this in several steps. Let us first assume that  $(A, \psi)$  is a Heisenberg group, i.e.,  $Z(A)$  is trivial. In this case the map  $e$  descends to a dense embedding  $A/B \hookrightarrow \hat{B}^\perp$ , which therefore acts as a natural “thickening” of the homogeneous space  $A/B$  to support a quasi-invariant measure. Indeed we can embed  $B^\perp$  diagonally in the Heisenberg group  $\tilde{A} \times_{\mathbb{T}} (\tilde{B}^\perp/B)_{op}$  as a Lagrangian subgroup. For this group,

$$V\left(\tilde{A} \times_{\mathbb{T}} (\tilde{B}^\perp/B)_{op}\right) \cong (V(A) \times V(A)_{op}) // \text{Lie}(B),$$

and therefore, by Proposition 1.14, a polarization of  $A$  determines a polarization of this Heisenberg group, and by Theorem A.5 specifies a unique irreducible representation. Applying Proposition A.7, this determines a measure on  $\hat{B}^\perp$ .

The general case is slightly more difficult. Again we use  $\widehat{A/B}$  as a thickening of the homogeneous space  $A/B$  to support a measure. When  $A$  is merely a generalized Heisenberg group, the map  $e$  defines an isomorphism

$$\widehat{A/B} \cong (B^\perp / (\widehat{Z(A)} \cap B)).$$

We now have an exact sequence

$$0 \longrightarrow (\widehat{Z(A)} \cap B) \longrightarrow \widetilde{B^\perp} \xrightarrow{\text{diag}} \tilde{A} \times_{\widehat{Z(A)}} (\tilde{B}^\perp/B)_{op},$$

where the image of  $B^\perp$  is isotropic, even Lagrangian. The group on the right is a generalized Heisenberg group with center equal to

$$Z(A, B) := Z(A) / (Z(A) \cap B).$$

Again, the polarization of  $A$  induces a polarization of this generalized Heisenberg group and it has a standard representation on the Hilbert space

$$(A.5) \quad \int_{\widehat{Z(A, B)}} \mathcal{H}_{A, (\zeta, \chi)} \otimes \mathcal{H}_{B^\perp, \chi}^* d\mu(\zeta),$$

where  $d\mu$  is the Haar measure on  $\widehat{Z(A, B)}$ . By the spectral theorem we now have:

**Proposition-Definition A.8.** Let  $(B, \chi) \subset (A, \psi)$  be as above. A polarization class of  $A$  defines a unique measure class  $\mu$  on  $\widehat{A/B}$ , quasi-invariant under the action of  $A$ , which defines the induced representation as

$$\text{Ind}_B^{\tilde{A}}(\mathbb{C}_\chi) := L^2\left(\widehat{A/B}, d\mu^\chi\right).$$

**Remark A.9.** A priori, only the measure class is determined by the polarization class. However, in the cases needed in the paper, a polarization determines a cyclic vector  $\Omega$ , unique up to phase, in the Hilbert space (A.5), cf. Proposition 1.10. By the above, the polarization then determines by Bochner's theorem a unique quasi-invariant measure in the class which defines the induced representation.

Let us first notice the following:

**Proposition A.10.** When  $A$  and  $B$  are finite dimensional, the definition above coincides with the usual definition of an induced representation.

*Proof.* In this case, the nondegenerate pairing introduced above defines an isomorphism  $\widehat{A/B} \cong A/B$ , and the induced measure is equivalent to the unique  $A$ -invariant measure on  $A/B$  induced by the Haar measure on  $A$ . With this, it is not difficult to see that the definitions coincide.  $\square$

On the other hand, the construction of the measure immediately show the following:

**Theorem A.11.** Let  $(B, \chi) \subset A$  as above. The induced representation  $\text{Ind}_B^{\tilde{A}}(\mathbb{C}_\chi)$  extends to a representation of  $\tilde{A} \times \tilde{B}_{op}^\perp$ : there is an isomorphism

$$\text{Ind}_B^{\tilde{A}}(\mathbb{C}_\chi) \cong \int_{\widehat{Z(A, B)}}^{\oplus} \mathcal{H}_{A, (\zeta, \chi)} \otimes \mathcal{H}_{B, \zeta}^* d\mu(\zeta),$$

where  $\hat{\mu}$  is a Haar measure on  $\widehat{Z(A, B)}$ .

**Corollary A.12.** When  $B$  is Lagrangian, i.e., maximally isotropic, the induced representation  $\text{Ind}_B^{\tilde{A}}(\mathbb{C}_\chi)$  is irreducible.

*Proof.* Notice that because  $B$  is maximal isotropic, it must contain the center  $Z(A)$ . Furthermore  $B^\perp = B$ , and the result follows from the previous theorem.  $\square$

Finally, using the diagonal embedding  $A \hookrightarrow \tilde{A} \times \tilde{A}_{op}$  we find the following ‘‘Plancherel formula’’:

**Proposition A.13.** Let  $(A, \psi)$  be a generalized Heisenberg group. A polarization determines a unique quasi-invariant measure-class  $\mu$  on  $\hat{A}$  for which we have

$$L^2(\hat{A}, d\mu) \cong \int_{\widehat{Z(A)}} \mathcal{H}_{A, \chi} \otimes \mathcal{H}_{A, \chi}^* d\hat{\mu}(\chi),$$

where  $\hat{\mu}$  denotes the Haar measure on  $\widehat{Z(A)}$ .

**Example A.14.** Consider again the level  $\ell$  central extension of the loop group. The canonical polarization class determines a measure  $\mu^\ell$  on the Pontryagin dual of  $LT$  and in this case we find

$$L^2(\widehat{LT}; d\mu^\ell) \cong \bigoplus_{\varphi \in \hat{A}} \mathcal{H}_\varphi \otimes \mathcal{H}_\varphi^*.$$

This is a version of the Peter–Weyl theorem for loop groups.

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